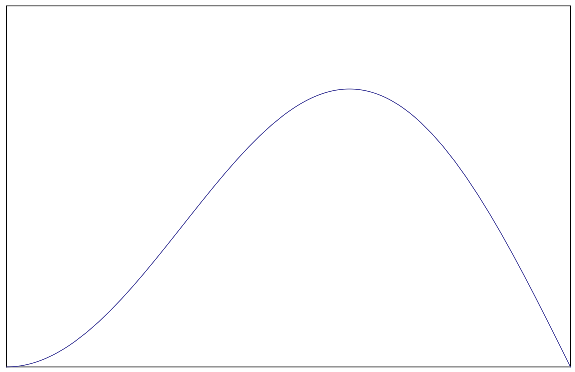
Freshman-Sophomore Contest 2012

Solutions, Second-Year Student Version

1. Let $f(x) = \sin x \sin 2x$.
(**a**) Sketch the graph of $f(x)$ on the interval $0 \le x \le \pi/2$.



(**b**) Find $\int_{x=0}^{\pi/2} f(x)\,dx$. Note that $\sin 2x = 2\sin x\cos x$. Thus with the substitution $u = \sin x$, $du = \cos x\,dx$, the original integral works out to $\int_{u=0}^{1} 2u^2\,du = 2/3$.
(**c**) Find the maximum of $f(x)$ on the interval $0 \le x \le \pi/2$. The original function has derivative $2\sin x\cos^2 x - \sin^3 x$ and using $\cos^2 x = 1 - \sin^2 x$ this simplifies to $2\sin x - 3\sin^3 x$. The maximum does not occur at the ends of the interval because the function is zero there and positive in between, and f is differentiable, so the maximum occurs at some point in the open interval $(0, \pi/2)$ at which $2\sin x - 3\sin^3 x = 0$. Since $\sin x \ne 0$ in this interval, we must have $2 - 3\sin^2 x = 0$. That means $\sin x = \sqrt{2/3}$, so $\cos x = \sqrt{1/3}$, so $f(x) = 2(2/3)/\sqrt{3}$. Tidied up, that's $(4/9)\sqrt{3}$, which is the answer.

2. Let $g(t) = \exp(-t^2/2)$ and let $G(x) = \int_{t=0}^{x} g(t)\,dt$.
(**a**) Give the Taylor's series expansion of $g(t)$ about zero, that is, write $g(t)$ in the form

$$g(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,$$

and give a formula for the general coefficient a_n in terms of n. The power series expansion for e^z is $e^z = 1 + z + z^2/2! + z^3/3! + \cdots$. Taking $z = -(1/2)t^2$ gives $g(t) = 1 + 0t - (1/2)t^2 + 0t^3 + (1/2!)(1/2)^2 t^4 + 0t^5 - (1/3!)(1/2)^3 t^6 + \cdots$. The coefficient on t^{2m+1} is zero, and the coefficient on t^{2m} is $(-1)^m 2^{-m}/m!$.
(**b**) Likewise, give the Taylor's series expansion of $G(x)$ about zero. The series expansion of an integral is the term by term integral of the series expansion of the function that was integrated. Here, that means that

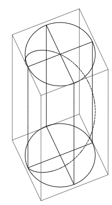
$G(x) = x - (1/6)x^3 + (1/40)x^5 - (1/336)x^7 + \cdots$; the coefficient on x^{2m} is zero and the coefficient on x^{2m+1} is $(-1)^m 2^{-m}/(2m+1)m!$.
(**c**) Find a decimal approximation to $G(1/2)$, accurate to within $\pm 10^{-3}$. (Of course, $G(1/2)$ is not equal to π, but if it were, you'd answer 3.142.) Only the first three terms are needed, because after that the terms are much less than 10^{-3} and since the sum is alternating with terms decreasing, the absolute value of the first term left out gives an upper bound for the error. The arithmetic is that $1/2 = 0.5000$, $1/48 \approx 0.0208$, and $1/1280 \approx 0.0008$. Adding and subtracting terms gives 0.4800 which is more than sufficient for three places: the answer is 0.480. With a computer, more digits can be got easily enough along the same lines; for instance, 0.4799252190.

3. Consider the sequence (c_n) given for $n \geq 2$ by $c_n = 2\ln(n) - \ln(n^2 - 1)$. (Thus, $c_2 = 2\ln 2 - \ln 3$.) Find, with proof, $\sum_{n=2}^{\infty} a_n$. This is a telescoping sum. First, observe that $\ln(n^2 - 1) = \ln((n-1)(n+1)) = \ln(n-1) + \ln(n+1)$. Next, look at the first several terms from this perspective:

$$2\ln 2 - \ln 1 - \ln 3 + 2\ln 3 - \ln 2 - \ln 4 + 2\ln 4 - \ln 3 - \ln 5$$
$$+ 2\ln 5 - \ln 4 - \ln 6 + 2\ln 6 - \ln 5 - \ln 7 + \cdots ;$$

Apart from one of the original $\ln 2$'s, everything cancels. The value is $\ln 2$, or numerically, about 0.69. A more rigorous solution would go into why the sum converges. If one quits at the term $2\ln n - \ln(n-1) - \ln(n+1)$, then the partial sum, ending with that term, has mostly canceled, with $\ln 2 + \ln n - \ln(n+1)$ left over. Since $\ln n - \ln(n+1) = \ln(n/(n+1))$ and $n/(n+1)$ tends to 1 as n tends to infinity, and since $\ln 1 = 0$ and the log function is continuous, this means that $\ln n - \ln(n+1)$ goes to zero so the partial sums converge to $\ln 2$ as n tends to infinity.

4. An ant walks along one of the spirals on a barber shop pole. The pole is a cylinder of radius 10 cm and the spiral makes an angle of 45 degrees with each (vertical) line on the cylinder. The ant starts at a certain point on the cylinder and climbs until it's reached a point directly above the point at which it started. It crawls at a uniform speed.



Find the average value of the square of the straight line distance, through the cylinder, from the ant's starting point to its current position, during the ant's walk. The ant's path can be parametrized as $x = 10\cos t$, $y = 10\sin t$, $z = 10t$, with t running from 0 to 2π. The vector from $(10, 0, 0)$ to the ant's position at time t is then $(10\cos t - 10, 10\sin t, 10t)$ and the square distance is $100((\cos t - 1)^2 + \sin^2 t + t^2)$. Integrating over t from 0 to 2π and dividing by 2π gives

$$\frac{100}{2\pi}\int_{t=0}^{2\pi}(1 - 2\cos t + \cos^2 t + \sin^2 t + t^2)\,dt = 200\left(1 + \frac{2}{3}\pi^2\right).$$

5. Find

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty}(1 + x^2 + y^2)^{-2}\,dy\,dx.$$

Going into polar, the integral becomes

$$\int_{r=0}^{\infty}\int_{\theta=\pi/4}^{\pi/2}\frac{r\,d\theta\,dr}{(1 + r^2)^2}.$$

The inner integral contributes a factor of $\pi/4$. With the change of variable $u = 1 + r^2$, the outer integral then becomes

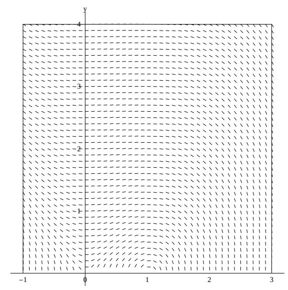
$$\frac{\pi}{4}\int_{u=1}^{\infty}\frac{1}{2}\frac{du}{u^2} = \frac{\pi}{8}.$$

6. Consider the differential equation

$$\frac{dy}{dx} = \frac{x - x^2}{y}$$

in the region $y > 0$.
(**a**) Sketch a direction field for this differential equation in the domain

$-1 \leq x \leq 3$, $0 \leq y \leq 4$.



(**b**) There is a solution $y = \phi(x)$ to the differential equation, passing through a point $(1, c)$, such that $\lim_{x \to 3^-} \phi(x) = 0$. Find the (algebraic) formula for $\phi(x)$ and in particular, find $c = \phi(1)$. In other words, if $y = 0$ at $x = 3$, what was y at $x = 1$?

The DE is separable. One gets $y\,dy = (x - x^2)\,dx$, thus $\frac{1}{2}y^2 = \frac{1}{2}x^2 - \frac{1}{3}x^3 + C$. When $x = 3$, $y = 0$ so $C = 9/2$. Thus $y = \sqrt{x^2 - (2/3)x^3 + 9}$, and in particular, when $x = 1$, $y = \frac{2}{3}\sqrt{7}$.