

THE MAZE: SOLUTIONS

The Maze consists of passages that connect chambers and do not intersect with each other. The Maze is flat. Exactly three passages meet at each chamber. Only one passage leads into the Maze from the outside. The Maze may have no loops: each passage connects different chambers. The Maze may have no double passages: no two passages connect the same two chambers. All chambers can be reached from the entrance.

- (1) (10 pt) Find the smallest possible number of chambers in the Maze. Provide proof and example.

Solution: Note that the number of chambers is always odd. Indeed, if there are N chambers, the number of passages is $(3N + 1)/2$ since three passages meet at each chamber and every passage except the entrance passage connects two chambers. This should be integer, hence N is odd.

The entrance chamber leads to chamber 1, that is connected to two more different chambers, 2 and 3. The Maze cannot contain three chambers only, otherwise there will be at most 2 passages that connect chamber 2 to 3 and 1. Since the number of chambers is odd, the smallest possible number of chambers is 5.

Below is an example of a Maze with 5 chambers.

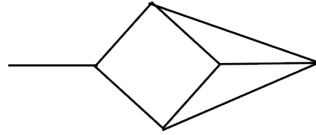


FIGURE 1. Maze with 5 chambers

- (2) (10 pt) Select integer n and prove that for all odd $C \geq n$, there exists a Maze with exactly C chambers, all passages straight and of same length, and all angles between passages being multiples of 60 degrees.

Team(s) with the smallest ~~n~~ in their proof get +5 pts.

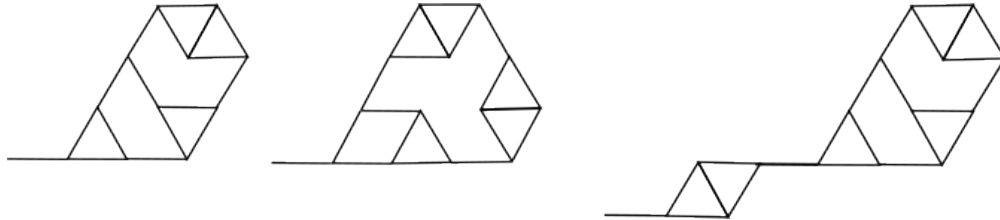


FIGURE 2. Mazes for problem (2) with 11, 13, and 15 vertices. The Maze with 15 vertices is obtained from the Maze with 11 vertices

Solution. We select $n = 11$. Below are examples of suitable Mazes with 11 and 13 chambers. Now, we can replace the entrance passage with a chain of pieces with 4 chambers as shown on Fig. 2. By adding 4 vertices to 11 or 13 repeatedly, we can get a Maze with any odd number of chambers that is greater than 9.

- (3) (10+10+10+10 pt) For the Maze shown above, the robot enters the Maze and chooses left or right passage with probability $1/2$ in each chamber. Each passage, including the entrance passage, takes 1 minute to walk through. Once the robot exits the Maze, it does not come back. Find the probability of the fact that the robot exits after (a) 5 minutes; (b) 7 minutes or less; (c) 8 minutes or less; (d) 10 minutes or less.

(Note that the robot cannot exit in less than 5 minutes).

At time 1, the robot arrives to the entrance chamber. Every minute, he chooses 2 options (left or right) with equal probability. Hence at time n , there are 2^{n-1} possible routes that the robot can take, and each has probability 2^{-n+1} .

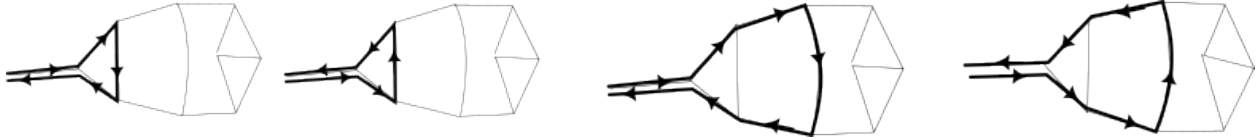


FIGURE 3. Routes of length 5 and 7

(a) There are only 2 routes of length 5 that exit the maze (see Fig. 3 left) and each has probability 2^{-4} , hence the probability is $2/2^4 = 1/8$.

(b) There is no way to exit in exactly 6 minutes and only 2 ways to exit in exactly 7 minutes (see Fig.3 right), hence the probability is $1/8 + 2/2^6 = 1/8 + 1/32 = 5/32$ (it includes probability from (a)).

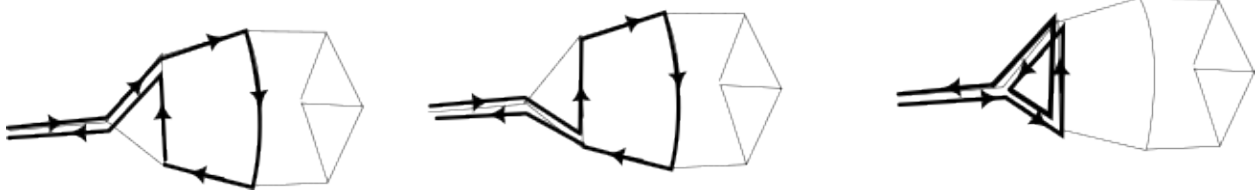


FIGURE 4. Routes of length 8

(c) There are 6 ways to exit the maze in 8 minutes (see Fig.4 for three of them; there are also 3 routes symmetric to those — note that the first two routes do not differ by symmetry only).

Hence the probability is $5/32 + 6/2^7 = 13/64$.

(d) There are two ways to exit in exactly 9 minutes (See Fig.5 left for one of them, the second one is symmetric); there are 8 ways to exit in exactly 10 minutes (4 are shown on Fig. 5 right, and there are 4 more that are symmetric).

Hence the probability is $13/64 + 2/2^8 + 8/2^9 = 29/128$.

Another way to solve all four items is to mark the probability for the robot to traverse the given edge in the given direction during each minute. See Fig.6.

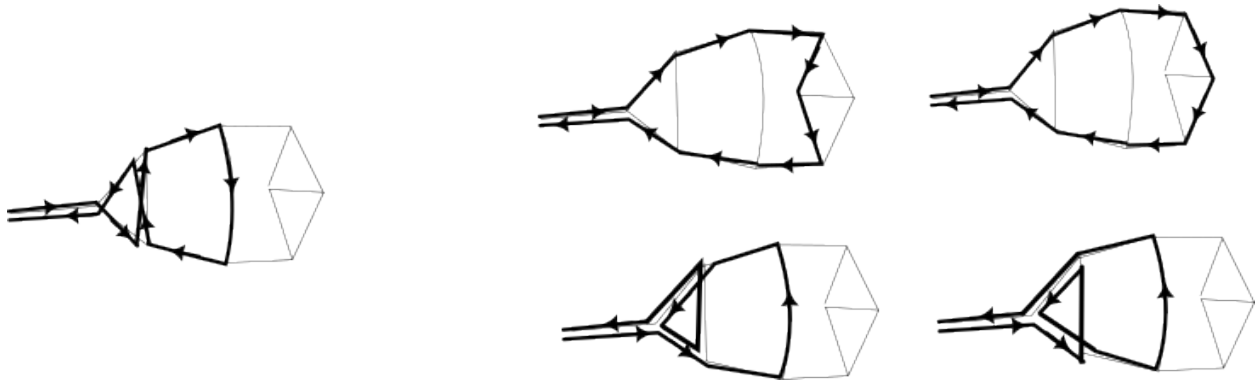


FIGURE 5. Routes of length 9 and 10

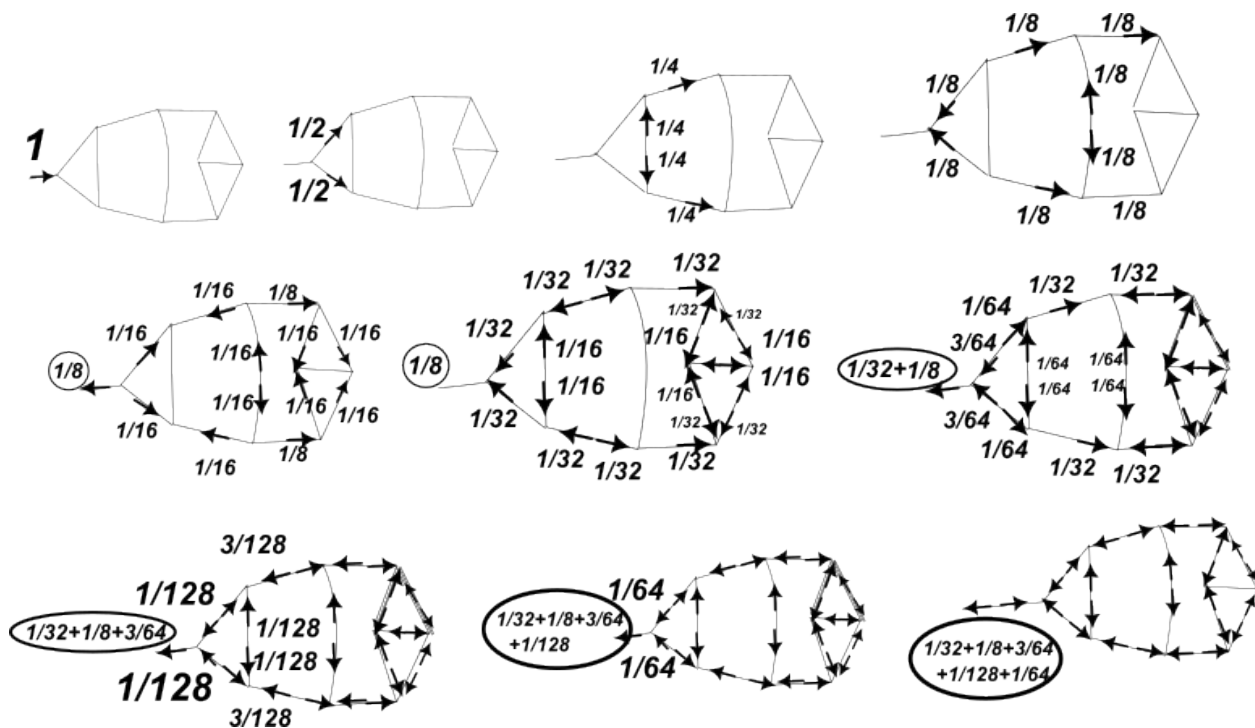


FIGURE 6. Probability of walking in each possible direction during 1st, 2nd, ...10th minute. We did not label arrows that are too far from the exit and cannot affect the answer.

- (4) (15 pt) The robot moves in the same way as in Problem (3). Show that for every Maze, the robot will eventually exit the Maze with probability 1.

Lemma: from any position in the passage of the Maze, the robot can exit.

This lemma is almost trivial and might be omitted in students' proofs, but we include the formal proof below.

Proof of the lemma: Let the robot be heading from chamber A to chamber B along the passage p . Find a way from B to the exit. If it does not start with the passage p , the robot can follow it after he goes to B and exit the Maze. However, the robot may not turn around, so we need to consider the case when all routes from B to the exit include p . In this

case, B is located in the piece P of the Maze that becomes inaccessible from the entrance if p is removed. Note that in this case, A must be accessible from the entrance if p is removed, and let w be the way from A to the exit. Find a loop l joining B to itself within this piece P (e.g. by turning right in each chamber until the robot returns to B). Then the robot can follow the loop l , the passage p , and then head to the exit from A following the way w .

Solution. Let δ_k be the probability of staying in the Maze after k minutes. Due to the Lemma, the robot can exit from any position in the passage of the Maze. If the length of the longest exiting route is n , for any moment of time the following is true: *from any position, the robot will exit in the next n minutes with probability at least $1/2^n$* . Hence the probability of staying in the Maze decreases by at least $\delta_k/2^n$ in the next n minutes: $\delta_{k+n} \leq \delta_k(1 - 1/2^n)$. Since n only depends on the Maze, the probability of staying in the Maze decreases in geometric progression, and tends to zero.

All passages in the Maze in problems (5 – 8) are straight, but can be of different length.

The maintenance team is going to mount lamps in some of the chambers so that all passages are lit (including the entrance passage). The passage is lit if there is a lamp in at least one of the two chambers that it joins. Let the number of chambers be C .

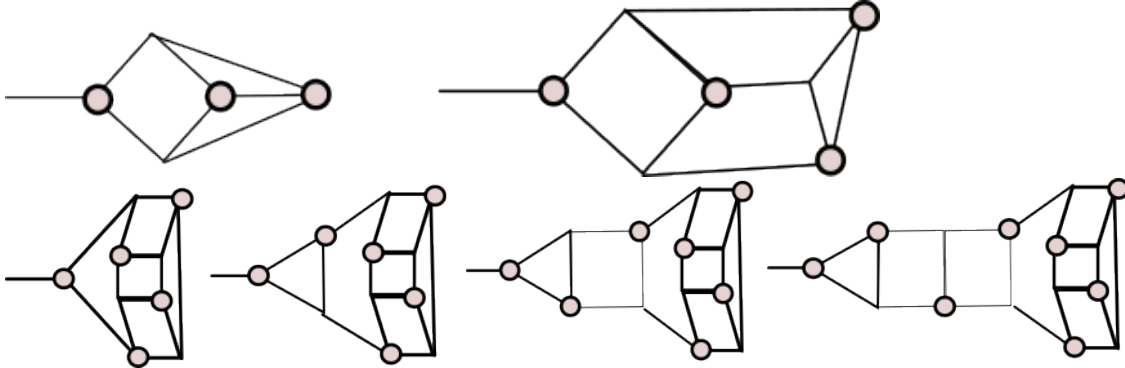
In problems 5 – 8, generous partial credit is given to teams that obtain slightly weaker estimates on the number of lamps, especially if these estimates only differ from requested estimates by constants independent on C . Teams must clearly state estimates that they are proving. In problems 6 and 8, teams that give examples for $C \geq N$ with some other N instead of 5 or 11 respectively, will also get generous partial credit.

- (5) (10 pt) Prove that every Maze with C chambers needs at least $(C + 1)/2$ lamps.

Solution. Indeed, n lamps will light up at most $3n$ passages while the total number of passages is $(3C + 1)/2$. Hence $3n \geq (3C + 1)/2$, i.e. $n \geq (C + 1/3)/2$. Since C is odd, $(C - 1)/2$ is integer and $n > (C - 1)/2$, hence $n \geq (C + 1)/2$.

- (6) (15 pt) For every odd $C \geq 5$, give an example of the Maze with C chambers such that $(C + 1)/2$ lamps are sufficient for it.

Solution. See below. The first picture shows Mazes with 5 and 7 chambers. The second picture shows Mazes with 9, 11, and 13 chambers; by increasing the number of squares, we get Mazes with any odd number of chambers with $C \geq 9$.



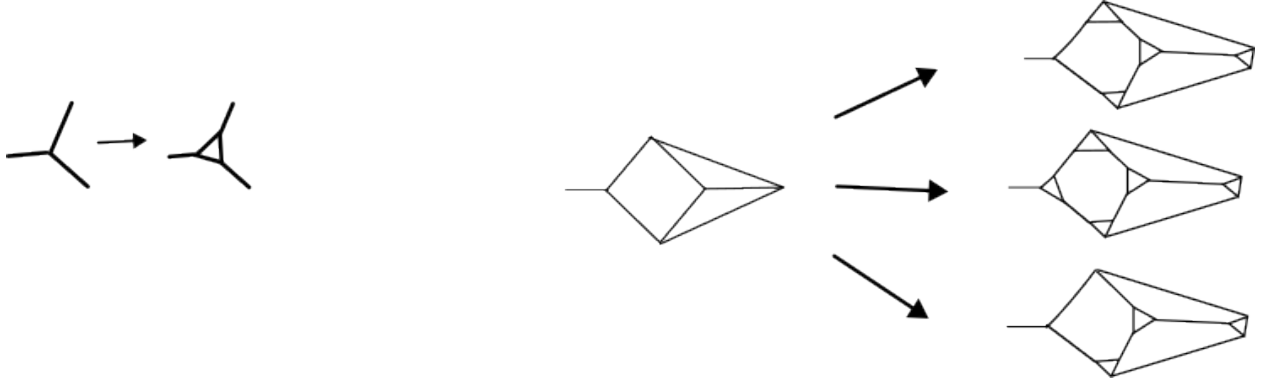


FIGURE 7. Pictures for Problem (8). Left: replacing a chamber with a triangle. Right: Mazes with 13, 15, and 11 chambers obtained from a Maze with 5 chambers

- (7) (10 pt) Prove that $\lfloor (3C + 1)/4 \rfloor$ lamps are sufficient for every Maze with C chambers.

Here, $\lfloor x \rfloor$ is the greatest integer less than or equal to x .

Solution. We will add lamps at all chambers at first, and then select chambers where we are going to remove lamps. If n chambers are already dark but all passages are lit, we may remove one more lamp in any chamber that is not an entrance chamber and is not connected by passages to any of these n dark chambers. This means that there are at most $3n + 1$ chambers where we may not remove lamps. If $n + 3n + 1 < C$, there will always be a chamber where we can remove the lamp.

Hence we may switch off $(C - 1)/4$ lamps if C has remainder 1 when divided by 4, and $(C + 1)/4$ lamps if C has remainder 3 when divided by 4. The number of remaining lamps is $(3C + 1)/4$ in the first case and $(3C - 1)/4$ in the second case, which is described by the formula $\lfloor (3C + 1)/4 \rfloor$ in any case.

- (8) (20 pt) For every odd $C \geq 11$, give an example of a Maze with C chambers that needs at least $\lfloor 2/3C \rfloor$ lamps.

Lemma: There exists a Maze with straight passages and any odd number of chambers $n \geq 5$.

Proof of the Lemma: see pictures for Problem (6).

Solution: Examples of Mazes with C chambers are constructed separately for different remainders of C modulo 3.

Remainder 1. Lemma guarantees that the Maze can have any odd number n of chambers with $n \geq 5$. In a Maze with n chambers, replace each chamber except the entrance chamber with a triangle, see Fig. 7. We get a Maze with $C = 3(n - 1) + 1$ chambers. Note that in each triangle, at least 2 out of 3 chambers must have lamps. Also, we must have a lamp in the entrance chamber. Hence we must have at least $2(n - 1) + 1 = 2/3(C - 1) + 1$ lamps. Since $2/3(C - 1) + 1 = 2/3C + 1/3 > \lfloor 2/3C \rfloor$, the statement follows.

Remainder 0. Similarly, we may replace all chambers with triangles, and get a Maze with $C = 3n$ chambers that needs $2n = 2/3C$ lamps.

Remainder 2. Finally, we can replace all but 2 chambers (entrance chamber and one more chamber) with triangles and get a Maze with $C = 3(n - 2) + 2$ chambers that requires $2(n - 2) + 1 = 2/3(C - 2) + 1$ lamps.

Since $2/3(C - 2) + 1 = \frac{2C-1}{3} = \lfloor 2C/3 \rfloor$, the statement follows.

The number of chambers in these Mazes is $3(n - 1) + 1$, $3n$, and $3(n - 2) + 2$ respectively, with $n \geq 5$. These are all odd numbers with $C \geq 11$.