Feasibility of p-adic Polynomials

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Motivation and applications

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- Cryptography
- Factoring rational polynomials
- Number theory

Motivation and applications

- Cryptography
- Factoring rational polynomials
- Number theory
- **Question**: given a system of polynomials over $\mathbb{Q}_p$, what are its roots?

What are the p-adics?

- Recall from real analysis that the real numbers $\mathbb{R}$ are the Cauchy sequence completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|$, where $|\cdot|$ is the usual absolute value, where the normal operations on $\mathbb{Q}$ are naturally extended to $\mathbb{R}$.

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- For a fixed prime number p, we can construct a metric from a different absolute value function. Note that for any nonzero rational number q, we can give it a unique prime factorization, allowing negative exponents (e.g., $28/9 = 2^2 3^{-2} 7^1$).

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- If p^k appears in the factorization of q, then we define the p-adic absolute value of q to be $|q|_p = p^{-k}$. (E.g., $|28/9|_2 = 2^{-2} = 1/4$, $|28/9|_3 = 3^2 = 9$, $|28/9|_5 = 5^{-0} = 1$.)

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- If p^k appears in the factorization of q, then we define the p-adic absolute value of q to be $|q|_p = p^{-k}$. (E.g., $|28/9|_2 = 2^{-2} = 1/4$, $|28/9|_3 = 3^2 = 9$, $|28/9|_5 = 5^{-0} = 1$.)
- If we additionally define $|0|_p = 0$, then the function $d_p(x, y) = |x - y|_p$ defines a metric. We define the p-adic numbers $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to this

Basic facts

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  where the a_i are integers between 0 and $p-1$. This is called the p-adic expansion of q.
- $\mathbb{Q}_p$ cannot be turned into an ordered field and is totally disconnected.

Polynomials over $\mathbb{Q}_p$

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- Systems of p-adic polynomials can be reduced to single polynomial equations through trickery. Over $\mathbb{R}$, it is easy to see that a system of polynomials $f_1, \ldots, f_n$ has a root if and only if the following polynomial has a root:

$$(f_1)^2 + (f_2)^2 + \cdots + (f_n)^2$$

- Something similar, but more complicated is possible over $\mathbb{Q}_p$

Polynomials over $\mathbb{Q}_p$

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- Two levels of complexity: number of variables and number of terms.
- Define $\mathcal{F}_{n,m}$ to be the set of polynomials in n variables and m terms.
  E.g., if $f(x, y) = 4 + 2x^{10}y^4 + x^{15}y^6$, then $f \in \mathcal{F}_{2,3}$.

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  E.g., if $f(x, y) = 4 + 2x^{10}y^4 + x^{15}y^6$, then $f \in \mathcal{F}_{2,3}$.
- But we can reduce further...

Honest polynomials

- Consider again $f(x, y) = 4 + 2x^{10}y^4 + x^{15}y^6$. We know that $f \in \mathcal{F}_{2,3}$.
- Substitute $z = x^5y^2$. Then f reduces to:

$$g(z) = 4 + 2z^2 + z^3$$

- Then we need only solve $g \in \mathcal{F}_{1,3}$. The set of solutions of f is then $\{(x, y) : x^5y^2 = z \text{ for some } z \text{ where } g(z) = 0\}$.

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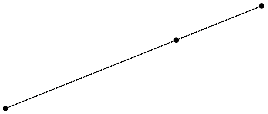
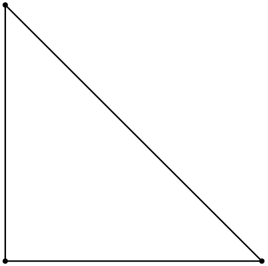
- Can this be done for other polynomials?

Honest polynomials

- $f(x, y) = 4x^0y^0 + 2x^{10}y^4 + x^{15}y^6$
- Each term has a factor of x and a factor of y; we can think of the exponents on each term as a vector in $\mathbb{Z}^2$, namely, $(0, 0), (10, 4), (15, 6)$ for the above three terms respectively.
- If we plot these points, they form a line. We call polynomials for which this happens *dishonest*, and all others *honest*.
- Example: $f(x, y) = 3 + x + 7y$ is honest, because its exponent vectors $(0, 0), (1, 0)$ and $(0, 1)$ do not lie in a line.
- We can always reduce dishonest polynomials to honest ones by change-of-variable methods as seen before. Thus, we can restrict study to honest polynomials. We denote by $\mathcal{F}^*_{n,m}$ the set of honest polynomials in n variables and m terms.



Newton polytope for $f(x,y) = 4x^0y^0 + 2x^{10}y^4 + x^{15}y^6$



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Simple cases

- Monomials
  - $x_1^{d_1} \cdots x_n^{d_n} = 0$, same as in real case (if and only if some $x_i = 0$ when $d_i x \neq 0$).

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- Monomials
  - $x_1^{d_1} \cdots x_n^{d_n} = 0$, same as in real case (if and only if some $x_i = 0$ when $d_i x \neq 0$).
- Binomials
  - $x^2 + 1 = 0$?

Hensel's Lemma

Theorem (Hensel's Lemma)

Let $f \in \mathcal{F}_{1,n}$, and suppose we have $x \in \mathbb{Q}_p$ such that:

- $f(x) \equiv 0 \bmod p$ *and*
- $f'(x) \not\equiv 0 \bmod p$.

Then there exists $x_0 \in \mathbb{Q}_p$ such that:

- $f(x_0) = 0$, *and*
- $x_0 \equiv x \bmod p$

- Uses adapted Newton's method
- Can be extended to higher powers of p, multiple variables

Hensel's lemma: example

- Consider $f(x) = x^2 + 1$ in $\mathbb{Q}_5$.

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- Consider $f(x) = x^2 + 1$ in $\mathbb{Q}_5$.
- $f(2) = 5 \equiv 0 \mod 5$ $f'(2) = 4 \not\equiv 0 \mod 5$, thus by Hensel's lemma, there exists an x_0 in $\mathbb{Q}_5$ satisfying $x_0^2 = -1$.

Hensel's lemma: example

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- Consider $f(x) = x^2 + 1$ in $\mathbb{Q}_5$.
- $f(2) = 5 \equiv 0 \mod 5$ $f'(2) = 4 \not\equiv 0 \mod 5$, thus by Hensel's lemma, there exists an x_0 in $\mathbb{Q}_5$ satisfying $x_0^2 = -1$.
- -1 has a square root! Which proves that $\mathbb{Q}_5$ cannot be ordered.

More complicated polynomials

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- We can apply a version of Hensel's lemma more generally.

Theorem (Birch and McCann)

Given a polynomial f in any number of variables over $[Q]_p$, there exists an integer $D(f)$ such that if for some x we have

$$|f(x)|_p < |D(f)|_p$$

then we can refine x to a true root of f. Moreover, we can calculate $D(f)$.

Birch and McCann

- Good news: we can, in finite time, check if a polynomial has a root. Just brute force check for:

$$f(x) \equiv 0 \mod p^R$$

where $p^R > |D(f)|_p^{-1}$.

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- Bad news: $D(f)$ is resource-intensive to calculate. If n is the number of variables and d the degree, then

$$L(D(f)) < (2^n dL(f))^{(2d)^{4^n} n!}$$

  - taking $f(x) = x^7 + 4$, we get:

$L(D(f)) <$ 1933326428133495947869005351579566442338794676438145065454226255104180462889721182185 7
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Feasibility of p-adic Polynomials

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More sophisticated results

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Theorem (Avendano, Ibrahim, Rojas, Rusek)

For a fixed prime p, finding a root to a function in $\mathcal{F}_{1,3}$ is **NP**. *Furthermore, allowing p to vary, finding roots for almost all polynomials in one variable with integer coefficients is* **NP**, *as it is for $\bigcup_n \mathcal{F}^*_{n,n+1}$.*

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- **NC** $\subseteq$ **P** $\subseteq$ **NP** $\subseteq$ **EXPTIME**
- For $f(x) = 4 + x^7$, taking $p = 2$, finding a root would require checking for a root to an associated polynomial over $\mathbb{Z}/32\mathbb{Z}$. Much better!

Neat result

- Define the p-adic Newton polytope of a polynomial $f(x) = a_0 x^{d_0} + \cdots + a_n x^{d_n}$ to be the convex hull of $\{(d_i, -\log_p |a_i|_p) : i = 0, \ldots, n\}$.
- It can be shown that if a lower edge of the p-adic Newton polytope has an inner normal vector of the form $(1, k)$ and horizontal length m, then f has exactly m roots with p-adic absolute value p^k in $\mathbb{C}_p$.

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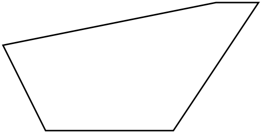
- Consider $f(x) = 1875 - (24875x)/4 + (24125x^2)/4 - (9605x^3)/4 + (1783x^4)/4 - 37x^5 + x^6$. Then the 2-adic Newton polytope tells us that there is one root with absolute value 1/4, two with absolute value 2, and ohe with absolute value 1. The roots of f are 20, 1/2,3/2, and 5 (with multiplicity three).



Acknowledgements

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