Visualizing $\mathcal{A}$ - Discriminants Chambers

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Algorithmic Algebraic Geometry REU

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Introduction

- We need a software package to visualize the $\mathcal{A}$-discriminant chambers when $\#\mathcal{A} = n + 4$
- Knowing the chambers helps us easily identify the topological type of the real zero set of real polynomials, among many other applications.

Projective Space

Let $\mathbb{K}$ be any field. We defined projective space $\mathbb{P}$ over the field $\mathbb{K}$ as follows:

$$\mathbb{P}^n_{\mathbb{K}} := \{[z_0 : \cdots : z_n] \mid z_i \in \mathbb{K} \textit{ not all zero}\}$$

with the identification

$$[z_0 : \cdots : z_n] = [z_0\lambda : \cdots : z_n\lambda] \textit{ for all } \lambda \in \mathbb{K}^*$$

Horn Kapranov Uniformization

Theorem (M. Kapranov)

Let $\mathcal{A} = \{a_1, \cdots, a_{n+k}\} \subset \mathbb{Z}^n$ *be the support for a given polynomial* f *and define the following matrix:*

$$\hat{\mathcal{A}} = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+k} \end{pmatrix}$$

The parametrization of $\nabla_{\mathcal{A}}$ *is given by the closure of*

$$\nabla_{\mathcal{A}} = \left\{ [\beta_1 \lambda_1 t^{a_1} : \cdots : \beta_{n+k} \lambda_{n+k} t^{a_{n+k}}] \mid \beta \in \mathbb{C}^{n+k},\ \hat{\mathcal{A}} B = 0,\ t \in (\mathbb{C}^*)^n \right\}$$

where β_i *are the rows of* B*.*

Example

$$f(x) = c_0x^7 + c_1x^4 + c_2x^5 + c_3x^3$$

$$\hat{\mathcal{A}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 7 & 4 & 5 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ -2 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\nabla_{\mathcal{A}} = \left\{ [\lambda_1 t^7 : 2\lambda_2 t^4 : (-2\lambda_1 - \lambda_2)t^5 : (\lambda_1 - \lambda_2)t^3] | \lambda_1, \lambda_2 \in \mathbb{C},\ t \in \mathbb{C}^* \right\}$$

Horn Kapranov Uniformization

You can define $\varphi : \mathbb{R}^{k-1} \to \mathbb{R}^{n+k}$ as $\varphi(\lambda) := \log|\lambda B^T|$

Corollary (Bastani, Hillar, Popov & Rojas)

$\log|\nabla_{\mathcal{A}}| = \log|\lambda B^T| +$ *row space of* $\hat{\mathcal{A}}$

Example

From...

$$\nabla_{\mathcal{A}} = \left\{[\lambda_1 t^7 : 2\lambda_2 t^4 : (-2\lambda_1 - \lambda_2)t^5 : (\lambda_1 - \lambda_2)t^3] | \lambda_1, \lambda_2 \in \mathbb{C},\ t \in \mathbb{C}^*\right\}$$

Taking log

$$\begin{aligned}\mathrm{Log}|\nabla_{\mathcal{A}}| = \{&(\log|\lambda_1|,\ \log|2| + \log|\lambda_2|,\ \log|2\lambda_1 + \lambda_2|,\ \log|\lambda_1 - \lambda_2|) + \\ &< (1,1,1,1) + (7,4,5,3) > \ |\ \lambda_1, \lambda_2 \in \mathbb{R}\}\end{aligned}$$

Reduction

Consider $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

$\frac{1}{c_0} f(x) = 1 + \frac{c_1}{c_0} x + \frac{c_2}{c_0} x^2 + \frac{c_3}{c_0} x^3$

$\frac{1}{c_0} f(\frac{c_0}{c_2} x) = 1 + \gamma x + x^2 + \Gamma x^3$

This can be generalizes using the B matrix:

$c^B = (c_0, c_1, c_2, c_3)^B = \left(\frac{c_0 c_3^2}{c_2^3}, \frac{c_1 c_3}{c_2^2} \right)$

Reduction

$Log|\nabla_{\mathcal{A}}| = Log|\lambda B^T| +$ row space of $\hat{\mathcal{A}}$

We want to consider... $Log|\bar{\nabla}_{\mathcal{A}}| = Log|\nabla_{\mathcal{A}}|B$

$\bar{\varphi} : \mathbb{P}_{\mathbb{R}}^{k-2} \setminus \mathcal{H}_B \to \mathbb{R}^{k-1}$ as $\bar{\varphi}(\lambda) := \log|\lambda B^T|B$

Reduction

$f(x) = c_0x^7 + c_1x^4 + c_2x^5 + c_3x^3$

From...

$\nabla_{\mathcal{A}} = \left\{[\lambda_1 t^7 : 2\lambda_2 t^4 : (-2\lambda_1 - \lambda_2)t^5 : (\lambda_1 - \lambda_2)t^3] | \lambda_1, \lambda_2 \in \mathbb{C},\ t \in \mathbb{C}^*\right\}$

Taking log

$$\begin{aligned}\mathrm{Log}|\nabla_{\mathcal{A}}| = \{&(\log|\lambda_1|,\ \log|2| + \log|\lambda_2|,\ \log|2\lambda_1 + \lambda_2|,\ \log|\lambda_1 - \lambda_2|)+ \\ &< (1,1,1,1) + (7,4,5,3) > \ |\ \lambda_1, \lambda_2 \in \mathbb{R}\}\end{aligned}$$

Reduction...

$$\begin{aligned}\log|\bar{\nabla}_{\mathcal{A}}| = \{&(\log|\lambda_1| + \log|\lambda_1 - \lambda_2| - 2\log|2\lambda_1 + \lambda_2|, \\ &2\log|2| + 2\log|\lambda_2| - \log|2\lambda_1 + \lambda_2| - \log|\lambda_1 - \lambda_2|)\ |\lambda_1, \lambda_2 \in \mathbb{R}\}\end{aligned}$$

Hyperplane Arrangement

Definition (Hyperplane)

A hyperplane $H \in \mathbb{R}^n$ is any set of the form

$$H = \{x \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = c\}$$

for $a \in \mathbb{R}^n$ and real number c.

Definition (Corresponding Hyperplane Arrangement)

$$\mathcal{H}_B = \{[\lambda] \in \mathbb{P}_{\mathbb{R}}^{k-2} \mid \lambda\beta_i = 0 \textit{ for some } i \in \{1, \cdots, n+k\}\}$$

$\bar{\varphi}(\lambda) = \log|\lambda B^T|B$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ -2 & -1 \\ 1 & -1 \end{pmatrix}$$

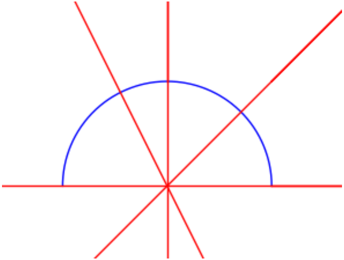
$$\mathcal{H}_B = \{[0:1],[1:0],[-2:-1],[1:-1]\}$$

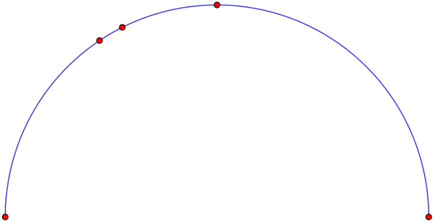
Lines through the origin in $\mathbb{R}^2$ where $\bar{\varphi}(\lambda)$ blow up to negative infinity.

n-variate $(n+3)$-case: example

Arrangement of points:

$$f(x) = c_1x^7 + c_2x^4 + c_3x^5 + c_4x^3$$

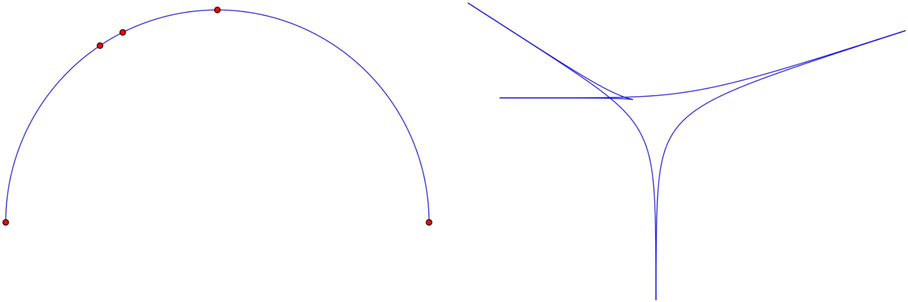




n-variate $(n+3)$-case: example

$\bar{\varphi}(\lambda) = \log|\lambda B^T|B$:

φ maps each arc segment to a piece of the contour amoeba



n-variate (n+4)-case

We are still working on developing a software package for the $(n+4)$-case.

Algorithm (outline)

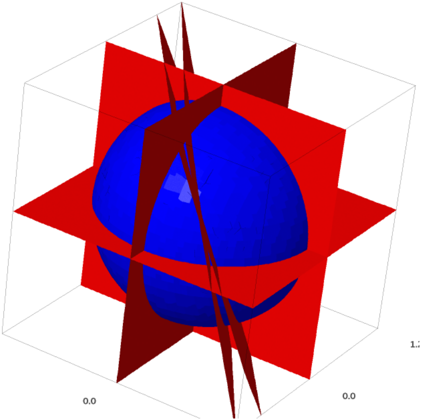
- Input: Support $\mathcal{A} \subset \mathbb{Z}^n$ with $\#\mathcal{A} = n + 4$
- Output: Surface of the reduced amoeba in $\mathbb{R}^3$

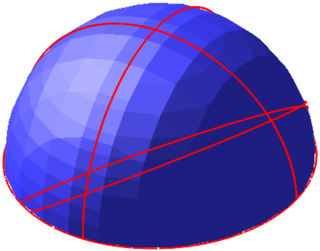
1. Basis for the right null-space of $\hat{\mathcal{A}}$
2. Determine the planes where $\bar{\varphi}(\lambda)$ blow up to infinity and represent it as circles (lines) in $\mathbb{P}^2$
3. Find the intersection lines between the planes and represent it as points in $\mathbb{P}^2$
4. Store all the information of vertices, edges and faces

n-variate (n+4)-case: example

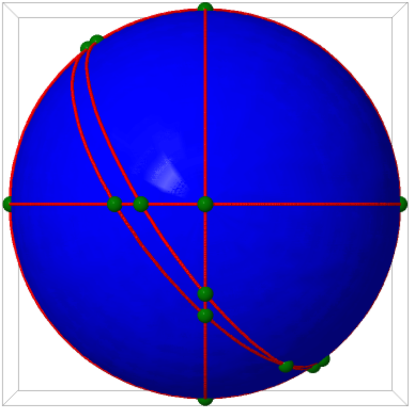
Arrangement of lines:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$





n-variate (n+4)-case: example



References

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Thank you for listening!