On classification of (weakly integral) modular categories by dimension

Kathleen Lee

July 20, 2017

Abstract

We look at classing strictly weakly integral modular categories of dimension $4q^2$ and 2^5. Motivation to classify these categories comes from their importance in various fields of mathematics, including topological quantum field theory, conformal field theory, representation theory of quantum groups, vertex operator algebras and applications in physics. To classify these categories we look at the possible number of invertible object and then look at the trivial component $\mathscr{C}_{ad}$. From there we can look at the cases for the non-integral component and classify the possibilities.

1 Introduction

A **modular category** is a non-degenerate pre-modular braided fusion category. This paper deals with the classification of **strictly weakly integral (s.w.i)** modular categories of dimension $4q^2$ with q being an odd prime and 2^5. A category is strictly weakly integral if it has components with non-integral dimension of the form $\mathbb{Z}\sqrt{n}$ with n a square free integer.

Theorem 1.1. *Let $\mathscr{C}$ be a s.w.i. modular category.*

1. *If $FPDim(\mathscr{C}) = 4q^2$ then $\mathscr{C}$ is*
   (a) *The gauging of the particle-hole symmetry of a pointed modular category of order q^2*
   or
   (b) *A generalized Tambara-Yamagami category*
2. *If $FPDim(\mathscr{C}) = 2^5$ then $\mathscr{C}$ is*
   (a) *$B \boxtimes I \boxtimes I$ where B is pointed with dimension 2 and I is an Ising modular category*
   or
   (b) *$I \boxtimes \mathscr{D}$ where $\mathscr{D}$ is pointed with dimension 2^3 and I is an Ising modular category*

2 Background

In this paper we deal with strictly weakly integral categories which have integer Frobenius-Perron dimension, where not all simple objects have integer dimensions [2]. The Frobenius-Dimension is the sum of the squared Frobenius-Dimension of the simple objects up to isomorphism classes [1].

From previous results we know that a weakly integral fusion category is faithfully graded by an elementary abelian 2-group [3]. Other useful results follow.

Theorem 2.1. *If p and q are odd primes and $\mathscr{C} \neq Vec$ is a fusion category of dimension $p^a q^b$ then $\mathscr{C}$ contains a non-trivial object [1].*

Generalized Tambara-Yamagami categories are non-pointed fusion categories where the tensor product of two non-invertible simple objects is a direct sum of invertible objects [3]. **Ising modular categories** are non-pointed modular categories with dimension 4 and rank 3 with two simple objects of dimension 1 and one of dimension $\sqrt{2}$ [2].

Let $\mathscr{D}$ be a full subcategory of $\mathscr{C}$. The **centralizer** of $\mathscr{D}$ in $\mathscr{C}$ is the full fusion subcategory

$$\mathscr{D}' := \{X \in \mathscr{C} | c_{Y,X} \circ c_{X,Y} = id_{X \otimes Y} \ \forall\ Y \in \mathscr{D}\} \ \ [1]$$

with the property that

$$\text{FPdim}(\mathscr{D}) \cdot \text{FPdim}(\mathscr{D}') = \text{FPdim}(\mathscr{C}).$$

If $\mathscr{C}$ is a weakly integral modular category with a unique simple isomorphism class of objects X such that $\text{FPdim}(X) \notin \mathbb{Z}$, then $\mathscr{C}$ is equivalent to an Ising modular category [2].

If the category $\mathscr{C}$ has the property that every every irreducible object X, where $\dim(X) \in \mathbb{Z}[\sqrt{2}]$, actually has dimension $\sqrt{2}$, then the category contains an Ising category.

A result that will be used in the 2^5 case:

Theorem 2.2. *If $\mathscr{C}$ is a s.w.i. modular category of dimension 2^4, then $\mathscr{C} = I \boxtimes \mathscr{D}$ where $\mathscr{D}$ is Ising or a pointed modular category with dimension 4 [3]*

another result that will be used:

Theorem 2.3. *If $\mathscr{C}$ is a s.w.i. modular category, then $4|dim(\mathscr{C})$ [3]*

3 Categories with dimension $4q^2$

Theorem 3.1. *Let $\mathscr{C}$ be a s.w.i. modular category with Frobenius-Perron dimension $4q^2$. Then $\mathscr{C}$ is*

1. *A gauging of $(\mathscr{C}_{int})_{\mathbb{Z}_2}$*

   or

2. A generalized Tambara-Yamagami category

Let $\mathscr{C}$ be a strictly weakly integral category of dimension $4q^2$. Since $\mathrm{FPdim}(x)^2$ must divide $4q^2$, the possible dimensions of the simple objects are $1, 2, q, 2q, \sqrt{2}, q\sqrt{2}, \sqrt{q}, 2\sqrt{q}$ and $\sqrt{2q}$. Recalling that there is an elementary abelian 2-group E that is a faithful grading, we can conclude that $|E|$ is either 2 or 4.

1. $|E| = 4$

   $\mathrm{FPDim}(\mathscr{C}_k) = \frac{FPDim(C)}{|E|} = q^2$. In this case looking at the integral part we have that

   $$q^2 = a + 4b + cq^2 + 4dq^2,$$

   where a, b, c and d are the number of simple objects of dimension $1,\ 2,\ q$ and $2q$ respectively.

   Since $4q^2 > q^2$ we have that $d = 0$. Given that a is the number of invertible elements and thus equal to $|U(C)|$, $4|a$, which also means $2|a$. But $a|q^2$ and by transitivity $2|q^2$, which is a contradiction because q is an odd prime. So $|E| \neq 4$.

2. $|E| = 2$

   $\mathrm{FPDim}(\mathscr{C}_k) = \frac{FPDim(C)}{|E|} = 2q^2$. Once again looking at the integral part we get that

   $$2q^2 = a + 4b + cq^2 + 4dq^2,$$

   where a, b, c and d are the number of simple objects of dimension $1,\ 2,\ q$ and $2q$ respectively. As in case 1, d is zero. Since $|E|$ must divide a and a must divide $\mathrm{FPDim}(\mathscr{C}_k)$, $2|a$ and $a|2q^2$ which means the number of invertible objects is $2,\ 2q$ or $2q^2$. We also note that $c = 0$, meaning in the integral component there are only invertible objects and objects of dimension 2.

   (a) $a = 2$

   When there are two invertible elements in $\mathscr{C}$, there are $\frac{1}{2}(q^2 - 1)$ simple objects of dimension 2. There are 4 possibilities for the non-integral component.

   Consider the category, B, generated by the non-trivial self-dual object. Either $g \in Z_2(B)$ or $g \notin Z_2(B)$.

   - $g \notin Z_2(B)$

     In this case $Z_2(B) = \mathrm{Vec}$, meaning B is a modular subcategory. Then $\mathscr{C} = B \boxtimes \mathscr{D}$ with $\mathscr{D}$ a strictly weakly integral category of dimension $2q^2$ but 4 must divide the dimension of $\mathscr{D}$. So this case cannot happen.
   - $g \in Z_2(B)$

- sVec
  In this case g cannot stabilize any object in the integral component. Since this is not the case, $Z_2(B)$ is not sVec.
- Tannakian
  In this case we can **de-equivariantize** $\mathscr{C}$ to C_G.
  If an object x is stabilized by g, then in C_G there are two objects with dimension $\frac{\mathrm{FPDim}(x)}{2}$.
  If an object y is mapped to an object w, then in C_G there is one object of dimension $\mathrm{FPDim}(y) = \mathrm{FPDim}(w)$.

i. q^2 simple objects with dimension $\sqrt{2}$
Since g stabilizes an objects of dimension $\sqrt{2}$ then in C_G there is an object with dimension $\sqrt{\frac{1}{2}}$ which cannot happen.

ii. 1 simple object with dimension $q\sqrt{2}$
Since this object is stabilized by g there is a simple object with dimension $\frac{q}{2}\sqrt{2}$ This is an improper dimension so this case cannot happen.

iii. q simple object with dimension $\sqrt{2q}$
Like in the previous cases a simple object with dimension $\sqrt{2q}$ is stabilized and in C_G there is an object with dimension $\sqrt{\frac{q}{2}}$ which is not a proper dimension.

iv. j simple object with dimension $2\sqrt{q}$ and $2(q-2j)$ simple objects dimension $\sqrt{q}$ with $j \leq \lfloor \frac{q}{2} \rfloor$.
Consider the object $\mathbb{1}$. Since $\mathbb{1} \otimes g = g$, meaning g does not stabilize 1, there is 1 invertible object in C_G.
Let Y_i be a simple object of dimension 2. Since $Y_i \otimes Y_i^* = \mathbb{1} \oplus g \oplus Y_k$ we know that g must stabilize all the simple objects of dimension 2. The $\frac{q^2-1}{2}$ simple objects of dimension 2 in $\mathscr{C}$ become $q^2 - 1$ invertible objects in C_G.
Let X_i be a simple object of dimension $2\sqrt{q}$. Again we see that $X_i \otimes X_i^* = \mathbb{1} \oplus g \oplus Y_K{}_{K=1}^{2q-1}$. Since g stabilizes X_i the j simple objects in $\mathscr{C}$ become $2j$ simple objects of dimension $\sqrt{q}$ in C_G.
Let Z_i be a simple object of dimension $\sqrt{q}$. Again we see that $Z_i \otimes Z_i^* = \mathbb{1} \oplus Y_K{}_{K=1}^{q-1}$. Since g does not stabilize Z_i the $2(q-2j)$ simple objects in $\mathscr{C}$ become $q-2j$ simple object of dimension $\sqrt{q}$ is C_G.
By collecting all the simple objects in C_G we get q^2 invertibles and q simple objects dimension $\sqrt{q}$. Since $(\mathscr{C}_{int})_{\mathbb{Z}_2}$ is pointed, $\mathscr{C}$ is a gauging of the particle-hole symmetry of a pointed modular category of order q^2.

(b) $a = 2q$
When there are $2q$ invertible objects and $\frac{1}{2}q(q-1)$ simple objects of dimension 2 there are three possibilities. First let us consider the case when C_{ad} has two invertibles and $\frac{q-1}{2}$ simple objects with

dimension 2. Like in the previous case if we look at B generated by the non-trivial self dual object, it must be Tannakian and we can then de-equivariantize the category to C_G.

i. $2q$ simple objects with dimension $\sqrt{q}$
Consider the object $\mathbb{1}$. Since $\mathbb{1} \otimes g = g$, meaning g does not stabilize $\mathbb{1}$, there is 1 invertible object in C_G.
Let Y_i be a simple object of dimension 2. Since $Y_i \otimes Y_i^* = \mathbb{1} \oplus g \oplus Y_k$ we know that g must stabilize all the simple objects of dimension 2. The $\frac{q^2-1}{2}$ simple objects of dimension 2 in $\mathscr{C}$ become $q^2 - 1$ invertible objects in C_G.
Let X_i be a simple object of dimension $\sqrt{q}$. Again we see that $X_i \otimes X_i^* = \mathbb{1} \oplus Y_K{}_{K=1}^{q-1}$. Since g does not stabilize X_i the $2q$ simple objects in $\mathscr{C}$ become q simple object of dimension $\sqrt{q}$ is C_G.
By collecting all the simple objects in C_G we get q^2 invertibles and q simple objects of dimension $\sqrt{q}$. Since $(\mathscr{C}_{int})_{\mathbb{Z}_2}$ is pointed, $\mathscr{C}$ is a gauging of the particle-hole symmetry of a pointed modular category of order q^2.

ii. q^2 simple objects with dimension $\sqrt{2}$.
Since an object of dimension $\sqrt{2}$ is stabilized by g in C_G, there is an object with dimension $\frac{\sqrt{2}}{2}$, which cannot happen.

iii. q simple objects with dimension $\sqrt{2q}$.
Like before all objects of dimension $\sqrt{2q}$ are stabilized by g and thus there exists an object with dimension $\frac{\sqrt{2q}}{2}$, which cannot happen.

(c) $a = 2q^2$
The only break down is when there are $2q^2$ invertible object and q^2 simple objects of dimension $\sqrt{2}$ in $\mathscr{C}$. Since the tensor product of any two non-invertible simple objects is a direct sum of invertible objects, $\mathscr{C}$ is a generalized Tambara-Yamagami category.

4 Categories with dimension 2^5

Theorem 4.1. *Let $\mathscr{C}$ be a s.w.i. modular category with Frobenius-Perron dimension 2^5. Then $\mathscr{C}$ is*

1. *$B \boxtimes I \boxtimes I$ where B is pointed with dimension 2 and I is an Ising modular category*

   or

2. *$I \boxtimes \mathscr{D}$ where $\mathscr{D}$ is pointed with dimension 2^3 and I is an Ising modular category.*

Let $\mathscr{C}$ be a strictly weakly integral category of dimension 2^5. Since $\mathrm{FPdim}(x)^2$ must divide 2^5, the possible dimensions of the simple objects are $1, 2, 2^2, \sqrt{2}, 2\sqrt{2}$, and $2^2\sqrt{2}$. Recalling that there is an elementary abelian 2-group E that is a faithful grading of $\mathscr{C}$, we can conclude that $|E| = 2$.

We can then find the dimension of the integral component, $\text{FPDim}(\mathscr{C}_k) = \frac{FPDim(C)}{|E|} = 2^4$. Once again looking at the integral part we get that

$$2^4 = a + 4b + 16c,$$

where a, b and c are the number of simple objects of dimension 1, 2 and 4 respectively. We know since $|E|$ must divide a and a must divide $\text{FPDim}(\mathscr{C}_k)$ that $2|a$ and $a|2q^2$ but since 4 must divide a we see that the number of invertible objects a is 2^2, 2^3 or 2^4. We also note that $c = 0$, meaning in the integral component there are only invertible objects and objects of dimension 2.

1. $a = 2^2$

   When there are two invertible elements in $\mathscr{C}$ there are 3 simple objects of dimension 2. There are 3 possibilities for the non-integral component. In $\mathscr{C}_{ad}$ there are 4 invertibles and 1 simple object of dimension 2

   Consider the category, B, generated by the non-trivial self-dual object in $\mathscr{C}_{ad}$, so $g \in Z_2(B)$ or $g \notin Z_2(B)$.

   Either

   - $g \notin Z_2(B)$
     In this case $Z_2(B) = \text{Vec}$, meaning B is a modular subcategory. Then $\mathscr{C} = B \boxtimes \mathscr{D}$ of $\mathscr{D}$ a strictly weakly integral category with dimension 2^4. This category is classified: $\mathscr{C} = I \boxtimes \mathscr{D}$ where $\mathscr{D}$ is Ising or pointed.
   - $g \in Z_2(B)$
     - sVec
       In this case g cannot stabilize any object in the integral component. Since this is not the case, $Z_2(B)$ is not sVec.
     - Tannakian
       In this case we can de-equivariantize $\mathscr{C}$ to C_G
       If an object x is stabilized by g, then in C_G there are two objects with dimension $\frac{\text{FPDim}(x)}{2}$.
       If an object y is mapped to an object w, then in C_G there is one object of dimension $\text{FPDim}(y) = \text{FPDim}(w)$.

   (a) 2 components with 1 simple object of dimension $2\sqrt{2}$

   In C_G we get that there are 8 invertibles and 4 simple objects of dimension $\sqrt{2}$ but this case cannot happen.

   (b) 1 component with 1 simple object dimension $2\sqrt{2}$ and 1 component with 4 simple objects dimension $\sqrt{2}$

   Since we cannot get a simple object of dimension $2\sqrt{2}$ from $I \boxtimes \mathscr{D}$ where $\mathscr{D}$ is Ising or pointed, this case cannot be modular. This case also cannot be Tannakian; by choosing the self-dual element that stabilizes a simple object of dimension $\sqrt{2}$ we get an object of dimension $\frac{\sqrt{2}}{2}$ which cannot happen.

(c) 2 components with 4 simple objects of dimension $\sqrt{2}$

By the self-dual element that stabilizes a simple object of dimension $\sqrt{2}$ we get an object of dimension $\frac{\sqrt{2}}{2}$, which cannot happen, so $\mathscr{C}$ is not Tannakian. It can be modular and since there is an object of dimension 2 we get that $\mathscr{C} = B \boxtimes I \boxtimes I$.

2. $a = 2^3$

When there are 2^3 invertible objects and 2 simple objects of dimension 2, there are three possibilities. Unlike the previous case we look at $\mathscr{C}_{int}$. In this case we find $Z_2(\mathscr{C}_{int})$ is equivalent to $\mathrm{Rep}(\mathbb{Z}_2) = < g >$ and Tannakian, allowing us to de-equivariantize to C_G. The only case for the non-integral component is 8 simple objects with dimension $\sqrt{2}$.

In this case g cannot stabilize any simple object of dimension $\sqrt{2}$. In C_G there are 8 invertibles and 4 simple objects of dimension $\sqrt{2}$. But by looking at this case further we see that it cannot happen.

3. $a = 2^4$

The only break down when there are 2^4 invertible objects is that there are 2^3 simple objects of dimension $\sqrt{2}$ in the non-integral components. Since the tensor product of any two non-invertible simple objects is a direct sum of invertible objects, it is a generalized Tambara-Yamagami category.

5 Discussion

Future work includes looking at $4p^2q$, $4p^2q^2$ and 2^n.

Acknowledgements

I would like to thank the National Science Foundation for funding this research experience. I would also like to thank Texas A&M University for hosting the research experience and my mentors, Dr. Julia Plavnik, Paul Gustafson and Ola Sobieska for giving guidance during the summer program, as well as my fellow REU participants.

References

[1] P. Bruillard, C. Galindo, S. Hong, Y. Kashina, D. Naidu, S. Natale, J. Plavnik, and E. Rowell, *Classification of integral modular categories of frobenius-perron dimension $pq^4 and p^2q^2$*, (2013).

[2] P. Bruillard, C. Galindo, S. Ng, J. Plavnik, E. Rowell, and Z. Wang, *On the classification of weakly integral modular categories*, (2015).

[3] P. Bruillard, J. Plavnik, and E. Rowell, *Modular categories of dimension p3m with m square-free*, (2017).