EFFECTIVE BOUNDS FOR TRACES OF MAASS-POINCARÉ SERIES

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Abstract. We give an asymptotic formula for the traces of Maass-Poincare series with an effective bound on the error term, and extend this to produce an effective upper bound for the traces themselves. These results can be specialized to the modular j-function. Zagier [8] has shown that the traces of the j-function appear as Fourier coefficients of weakly holomorphic modular forms of half-integral weight that are constructed using Zagier's lift. Further, works of Duke and Jenkins [1], and Miller and Pixton [4] show that the generating functions of traces of Maass-Poincare series appear in the holomorphic part of certain half-integral weight weakly holomorphic modular forms for $\Gamma_0(4)$. Thus our work can be used to give effective upper bounds for the Fourier coefficients of these half-integral weight modular forms.

1. Introduction and Statement of Results

Let $d < 0$ be an integer with $d \equiv 0, 1 \mod 4$. Let S_d be the set of $SL_2(\mathbb{Z})$-reduced, positive definite, integral binary quadratic forms of discriminant d, and let $h(d) = \#S_d$ be the class number of d. Given a form $Q(x, y) = a_Q x^2 + b_Q xy + c_Q y^2 \in S_d$, define the associated CM point

$$\tau_Q := \frac{-b_Q + \sqrt{d}}{2a_Q} \in \mathbb{H}$$

in the complex upper half-plane $\mathbb{H}$. Since Q is reduced, τ_Q lies in the standard fundamental domain for $SL_2(\mathbb{Z})$.

Let $f : \mathbb{H} \to \mathbb{C}$ be an $SL_2(\mathbb{Z})$-invariant function. Define the trace of $f(z)$ by

$$\mathrm{Tr}_d(f) := \sum_{Q \in S_d} f(\tau_Q).$$

In this paper we are interested in giving effective bounds for traces of a certain family of Maass-Poincaré series. To define these functions, for $\nu \in \mathbb{Z}^+$ and $s \in \mathbb{C}$, let

$$F_{s,\nu}(z) := 2\pi\nu^{s-\frac{1}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}\,(\gamma z)^{\frac{1}{2}} I_{s-\frac{1}{2}} \left(2\pi\nu \mathrm{Im}\,(\gamma z)\right) e\left(-\nu \mathrm{Re}\,(\gamma z)\right), \quad \mathrm{Re}(s) > 1$$

where $I_{s-1/2}$ is the I-Bessel function of order $s - 1/2$ and $e(z) := e^{2\pi i z}$. The Poincaré series $F_{s,\nu}(z)$ is an $SL_2(\mathbb{Z})$-invariant eigenfunction for the hyperbolic Laplacian Δ with eigenvalue $s(s-1)$. Moreover, Niebur [6] showed that $F_{s,\nu}(z)$ has an analytic continuation to $\mathrm{Re}(s) > 1/2$.

To motivate our interest in giving effective bounds for traces of the Poincaré series $F_{s,\nu}(z)$, we consider first the relationship between these functions and the classical modular j-function. Recall that the j-function is given by a Fourier expansion of the form

$$j(z) := q^{-1} + 744 + 196884 \cdot q + \cdots, \quad q := e(z).$$

The CM values $j(\tau_Q)$ are algebraic numbers called singular moduli. In the seminal work [8], Zagier proved that the generating functions of traces of singular moduli appear as holomorphic parts of half-integral weight weakly holomorphic modular forms for $\Gamma_0(4)$. It is known that

$$j(z) = F_{1,1}(z) + 720. \tag{1}$$

Therefore, it is natural to ask whether the generating functions of the traces of $F_{s,\nu}(z)$ also appear as holomorphic parts of half-integral weight weakly holomorphic modular forms for $\Gamma_0(4)$. This indeed the case, as is proved in the work of Duke and Jenkins [1], and Miller and Pixton [4]. Our Theorem 1 below can be used to give effective bounds for these Fourier coefficients.

Theorem 1. *Let $s \geq 1$ be a real number and $\nu \in \mathbb{Z}^+$. Then*

$$\mathrm{Tr}_d(F_{s,\nu}) = \sum_{Q \in S_d} 2\pi\nu^{s-\frac{1}{2}}(\mathrm{Im}(\tau_Q))^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi\nu\mathrm{Im}(\tau_Q))e(-\nu\mathrm{Re}(\tau_Q)) + E(s,\nu,d)$$

where $E(s,\nu,d)$ satisfies the bound

$$|E(s,\nu,d)| \leq C(s,\nu)h(d),$$

where $C(s,\nu)$ is the explicit function of s and ν defined by equation (39). In particular, we have

$$|\mathrm{Tr}_d(F_{s,\nu})| \leq C'(s,\nu)e^{\pi\nu\sqrt{|d|}}h(d).$$

where $C'(s,\nu)$ is an explicit constant that we define in equation (56).

Set $J(z) := j(z) - 744$ for convenience. Then recalling the identity (1), we immediately get the following corollary.

Corollary 1. *We have*

$$\mathrm{Tr}_d(J) = 2\pi \sum_{Q \in S_d} (\mathrm{Im}(\tau_Q))^{\frac{1}{2}} I_{\frac{1}{2}}(2\pi\mathrm{Im}(\tau_Q))e(-\mathrm{Re}(\tau_Q)) + E(1,1,d),$$

where

$$|E(1,1,d)| \leq (1.72 \times 10^6)h(d).$$

In particular, we have

$$|\mathrm{Tr}_d(J)| \leq (1.73 \times 10^6)h(d)e^{\pi\sqrt{|d|}}.$$

2. Effective bounds for the coefficients of $F_{\nu,s}(z)$

From [6] we have the following Fourier expansion for the Maass-Poincaré series:

$$F_{s,\nu}(z) = 2\pi\nu^{s-\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi\nu y)e(-\nu x) + \frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)}y^{1-s} + 4\pi\nu^{s-\frac{1}{2}}\sum_{n\neq 0} b(n,\nu;s)y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)e(nx)$$

where

$$b(n,\nu;s) := \sum_{c>0} \frac{S(n,-\nu;c)}{c} \begin{cases} I_{2s-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right) & n > 0 \\ J_{2s-1}\left(\frac{4\pi\sqrt{|n|\nu}}{c}\right) & n < 0, \end{cases}$$

$$S(a,b;c) := \sum_{\substack{d \mod c \\ (c,d)=1}} \left(\frac{a\bar{d} + bd}{c}\right)$$

is the ordinary Kloosterman sum, $\Gamma(s)$ is the gamma function, $\zeta(s)$ is the Riemann zeta function, $\sigma_{2s-1}(\nu)$ is the divisor function, and I_r, J_r, and K_r are the I, J, and K Bessel functions, respectively, of order r.

Proposition 1. *For any real number $s \geq 1$ we have the following bounds for $|b(n,\nu;s)|$. For $n \in \mathbb{Z}^-$:*

$$|b(n,\nu;s)| \leq C_1(s,\nu)\,|n|^s$$

and for $n \in \mathbb{Z}^+$:

$$|b(n,\nu;s)| \leq C_2(s,\nu) n^s e^{4\pi\sqrt{n\nu}},$$

where $C_1(s,\nu)$ and $C_2(s,\nu)$ are explicit functions of s and ν, given by equations (12) and (28), respectively.

Proof. Let $s \in \mathbb{R}$ be such that $s \geq 1$. First, using the series [7, 10.2.2]

$$J_{2s-1}(z) = \left(\frac{z}{2}\right)^{2s-1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z^2}{4}\right)^k}{k!\Gamma(2s+k)}$$

we find that

$$|J_{2s-1}(z)| \leq \left(\frac{z}{2}\right)^{2s-1} \sum_{k=0}^{\infty} \frac{1}{4^k k!\,|\Gamma(2s+k)|} \tag{2}$$

for $0 < z < 1$. If $z \geq 1$ then it is convenient to use the asymptotic formula [5, p. 3],

$$J_{2s-1}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left(\cos\omega \left(\sum_{k=0}^{N-1} (-1)^k \frac{a_{2k}(2s-1)}{z^{2k}} + R_{2N}^{(J)}(z,2s-1)\right) - \sin\omega \left(\sum_{m=0}^{N-1} (-1)^m \frac{a_{2m+1}(2s-1)}{z^{2m+1}} - R_{2N+1}^{(J)}(z,2s-1)\right)\right)$$

where $N := \left\lceil s - \frac{1}{2} \right\rceil$ and $\omega := z - \pi\left(s - \frac{1}{2}\right)$.
Note for the following inequalities that given $w \neq 0$ (see [5, p. 4])

$$\Lambda_p(w) := w^p e^w \Gamma(1-p,w)$$
$$\Pi_p(w) := \frac{1}{2}\left(\Lambda_p\left(we^{\frac{\pi}{2}i}\right) + \Lambda_p\left(w^{-\frac{\pi}{2}i}\right)\right),$$

where $\Gamma(1-p,w)$ is the incomplete Gamma function.
Now, from Theorem 1.5 in [5]

$$\left|R_{2N}^{(J)}(z,2s-1)\right| \leq \frac{|a_{2N}(2s-1)|}{|z|^{2N}} \sup_{r\geq 1} |\Pi_{2N}(2zr)|$$

and

$$\left|R_{2N+1}^{(J)}(z,r)\right| \leq \frac{|a_{2N+1}(2s-1)|}{|z|^{2N+1}} \sup_{r\geq 1} |\Pi_{2N+1}(2zr)|\,.$$

From [5, Proposition B.1], for $p > 0$,

$$\Pi_p(2zr) \leq 1. \tag{3}$$

Utilizing (3) and the fact that $z \geq 1$,

$$\left|R_{2N}^{(J)}(z,2s-1)\right| \leq |a_{2N}(2s-1)| \tag{4}$$

and

$$\left|R_{2N+1}^{(J)}(z,r)\right| \leq |a_{2N+1}(2s-1)|\,. \tag{5}$$

Furthermore, again since $z \geq 1$,

$$\left|\sum_{k=0}^{N-1}(-1)^k \frac{a_{2k}(2s-1)}{z^{2k}}\right| \leq \sum_{k=0}^{N-1} |a_{2k}(2s-1)| \tag{6}$$

and

$$\left|\sum_{m=0}^{N-1}(-1)^m \frac{a_{2m+1}(2s-1)}{z^{2m+1}}\right| \leq \sum_{m=0}^{N-1} |a_{2m+1}(2s-1)|. \tag{7}$$

Thus using inequalities (4)-(7) and the fact that $|\cos\omega| \leq 1$ and $|\sin\omega| \leq 1$,we get:

$$\begin{aligned}
|J_{2s-1}(z)| &\leq \left(\frac{2}{\pi z}\right)^{1/2}\left(|\cos\omega|\left(\sum_{k=0}^{N-1}\left|\frac{a_{2k}(2s-1)}{z^{2k}}\right| + \left|R_{2N}^{(J)}(z,2s-1)\right|\right)\right.\\
&\qquad\left. + |\sin\omega|\left(\sum_{m=0}^{N-1}\left|\frac{a_{2m+1}(2s-1)}{z^{2m+1}}\right| + \left|R_{2N+1}^{(J)}(z,2s-1)\right|\right)\right)\\
&\leq \left(\frac{2}{\pi z}\right)^{1/2}\left(\sum_{k=0}^{N-1}|a_{2k}(2s-1)| + |a_{2N}(2s-1)| + \sum_{m=0}^{N-1}|a_{2m+1}(2s-1)| + |a_{2N+1}(2s-1)|\right)\\
&= \left(\frac{2}{\pi z}\right)^{1/2}\sum_{k=0}^{2N+1}|a_k(2s-1)|
\end{aligned} \tag{8}$$

for $z \geq 1$. Recall the formula for $b(n,\nu;s)$ for $n<0$ is given by

$$b(n,\nu;s) = \sum_{c>0}\frac{S(n,-\nu;c)}{c}J_{2s-1}\left(\frac{4\pi\sqrt{|n|\,\nu}}{c}\right).$$

Using the Weil bound,

$$|S(n,-\nu;c)| \leq \tau(c)(n,-\nu,c)^{1/2}c^{1/2} \tag{9}$$

where τ is the divisor function, and noting that

$$(n,-\nu,c)^{1/2} \leq |n|^{1/2}$$

we get

$$|b(n,\nu;s)| \leq \sum_{c>0}\frac{\tau(c)\,|n|^{1/2}}{c^{1/2}}\left|J_{2s-1}\left(\frac{4\pi\sqrt{|n|\,\nu}}{c}\right)\right|.$$

We now want to split up the sum and use inequalities (2) and (8):

$$\begin{aligned}
|b(n,\nu;s)| &\leq |n|^{1/2}\left(\sum_{c=1}^{\left\lfloor 4\pi\sqrt{|n|\nu}\right\rfloor}\frac{\tau(c)}{c^{1/2}}\left|J_{2s-1}\left(\frac{4\pi\sqrt{|n|\,\nu}}{c}\right)\right| + \sum_{c>\left\lfloor 4\pi\sqrt{|n|\nu}\right\rfloor}\frac{\tau(c)}{c^{1/2}}\left|J_{2s-1}\left(\frac{4\pi\sqrt{|n|\,\nu}}{c}\right)\right|\right)\\
&\leq |n|^{1/2}\left(\left(\frac{1}{2\pi^2\sqrt{|n|\,\nu}}\right)^{1/2}\sum_{k=0}^{2N+1}|a_k(2s-1)|\sum_{c=1}^{\left\lfloor 4\pi\sqrt{|n|\nu}\right\rfloor}\tau(c)\right.\\
&\qquad\left. + \left(2\pi\sqrt{|n|\,\nu}\right)^{2s-1}\sum_{k=0}^{\infty}\frac{1}{4^k k!\,|\Gamma(2s+k)|}\sum_{c>\left\lfloor 4\pi\sqrt{|n|\nu}\right\rfloor}\frac{\tau(c)}{c^{2s-\frac{1}{2}}}\right).
\end{aligned}$$

Note that [2, p. 10] for all $\epsilon > 0$

$$\tau(c) \le C(\epsilon) c^{\epsilon}$$

where

$$C(\epsilon) := \prod_{p < 2^{1/\epsilon}} \max_{\alpha \ge 0} \frac{\alpha + 1}{p^{\alpha\epsilon}}.$$

Thus if we let $\epsilon = \frac{1}{2}$ we get that

$$\tau(c) \le C(1/2) c^{1/2} \tag{10}$$

noting that

$$C(1/2) := \frac{4\sqrt{6}}{e^2 \log 2 \log 3}.$$

It is also known that

$$\sum_{c>0} \frac{\tau(c)}{c^s} = \zeta^2(s) \tag{11}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Thus we use (10) and (11) to show that

$$\begin{aligned}
|b(n, \nu; s)| &\le |n|^{1/2} \left(\left(2\pi^2 \sqrt{|n|\,\nu}\right)^{-1/2} C(1/2) \sum_{c=1}^{\left\lfloor 4\pi\sqrt{|n|\nu} \right\rfloor} c^{1/2} + \left(2\pi \sqrt{|n|\,\nu}\right)^{2s-1} \sum_{k=0}^{\infty} \frac{1}{4^k k!\,|\Gamma(2s+k)|} \zeta^2\left(2s - \tfrac{1}{2}\right) \right) \\
&\le |n|^{1/2} \left((32\pi\,|n|\,\nu)^{1/2} C(1/2) \sum_{k=0}^{2N+1} |a_k(2s-1)| + |n|^{s-\frac{1}{2}} \left(2\pi\sqrt{\nu}\right)^{2s-1} \sum_{k=0}^{\infty} \frac{1}{4^k k!\,|\Gamma(2s+k)|} \zeta^2\left(2s - \tfrac{1}{2}\right) \right) \\
&\le \left((32\pi\nu)^{1/2} C(1/2) \sum_{k=0}^{2N+1} |a_k(2s-1)|) + (4\pi^2\nu)^{s-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{4^k k!\,|\Gamma(2s+k)|} \zeta^2\left(2s - \tfrac{1}{2}\right) \right) |n|^s.
\end{aligned}$$

Thus we see that if we define

$$\begin{aligned}
C_1(s, \nu) : &= \left((32\pi\nu)^{1/2} C(1/2) \sum_{k=0}^{2N+1} |a_k(2s-1)|) + (4\pi^2\nu)^{s-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{4^k k!\,|\Gamma(2s+k)|} \zeta^2\left(2s - \tfrac{1}{2}\right) \right) \\
&= \left((32\pi\nu)^{1/2} C(1/2) \sum_{k=0}^{2\left\lceil s-\frac{1}{2} \right\rceil + 1} |a_k(2s-1)|) + (4\pi^2\nu)^{s-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{4^k k!\,|\Gamma(2s+k)|} \zeta^2\left(2s - \tfrac{1}{2}\right) \right)
\end{aligned} \tag{12}$$

then

$$|b(n, \nu; s)| \le C_1(s, \nu)\, |n|^s$$

for $n < 0$, as stated in Proposition 1. We now bound $b(n, \nu; s)$ for $n > 0$. First, using the series [7, 10.25.2]

$$I_{2s-1}(z) = \left(\frac{z}{2}\right)^{2s-1} \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\,\Gamma(2s+k)}$$

we get that

$$|I_{2s-1}(z)| \leq \left(\frac{z}{2}\right)^{2s-1} \sum_{k=0}^{\infty} \frac{1}{4^k k! \, |\Gamma(2s+k)|} \tag{13}$$

for $0 < z < 1$. Now we will look at the case when $z \geq 1$. From [5, equation A.4] we get the identity

$$I_{2s-1}(z) = \mp \frac{i}{\pi} K_{2s-1}(ze^{\mp\pi i}) \pm \frac{i}{\pi} e^{\pm\pi i(2s-1)} K_{2s-1}(z) \tag{14}$$

and the asymptotic formulas

$$K_{2s-1}(ze^{\pm\pi i}) = \pm i \left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{M-1} (-1)^m \frac{a_m(2s-1)}{z^m} + R_M^{(K)}(ze^{\mp\pi i}, 2s-1)\right) \tag{15}$$

and

$$K_{2s-1}(z) = \left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z} \left(\sum_{m=0}^{M-1} \frac{a_m(2s-1)}{z^m} + R_M^{(K)}(z, 2s-1)\right) \tag{16}$$

where $M := \lceil 2s-1 \rceil$. From (14) we can write

$$|I_{2s-1}(z)| \leq \frac{1}{\pi}|K_{2s-1}(ze^{-\pi i})| + \frac{1}{\pi}|K_{2s-1}(ze^{\pi i})| + \frac{2}{\pi}|K_{2s-1}(z)|.$$

Now [5, equation 1.26] from Neme's Theorem 1.5 provides that for $|\arg w| < \frac{3\pi}{2}$ we get that

$$\left|R_M^{(K)}(w,\nu)\right| \leq \frac{|a_M(\nu)|}{|w|^M} \sup_{r\geq 1} \left|\Lambda_{M+\max(0,\frac{1}{2}-\nu)}(2wr)\right| \tag{17}$$

giving us that

$$\left|R_{\lceil 2s-1\rceil}^{(K)}(z, 2s-1)\right| \leq \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sup_{r\geq 1} \left|\Lambda_{\lceil 2s-1\rceil}(2zr)\right|$$

and

$$|R_{\lceil 2s-1\rceil}^{(K)}(ze^{\pi i}, 2s-1)| \leq \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sup_{r\geq 1} \left|\Lambda_{\lceil 2s-1\rceil}(2ze^{\pi i}r)\right|.$$

And then [5, Proposition B.1] tells us that

$$\left|\Lambda_{\lceil 2s-1\rceil}(2zr)\right| \leq 1, \tag{18}$$

and

$$\left|\Lambda_{\lceil 2s-1\rceil}(2ze^{\pm\pi i}r)\right| \leq \sqrt{e\left(\lceil 2s\rceil - \tfrac{1}{2}\right)}. \tag{19}$$

Thus from (18) and the fact that $z \geq 1$ we get that

$$\left|R_{\lceil 2s-1\rceil}^{(K)}(z, 2s-1)\right| \leq \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \tag{20}$$

and similarly, using (19)

$$|R_{\lceil 2s-1\rceil}^{(K)}(ze^{\pi i}, 2s-1)| \leq \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sqrt{e\left(\lceil 2s\rceil - \tfrac{1}{2}\right)}. \tag{21}$$

Also observe that as $z \geq 1$

$$\sum_{m=0}^{\lceil 2s-2\rceil} \left|\frac{a_m(2s-1)}{z^m}\right| \leq \sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)| \tag{22}$$

and

$$\sum_{m=0}^{\lceil 2s-2\rceil} \left|(-1)^m \frac{a_m(2s-1)}{z^m}\right| \leq \sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)|\,. \tag{23}$$

Using equation (16) and inequalities (20) and (22), we see that

$$\begin{aligned} |K_{2s-1}(z)| &\leq \left|\left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z}\right| \left(\sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right|\right) \\ &= \left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z} \sum_{m=0}^{\lceil 2s-1\rceil} |a_m(2s-1)|. \end{aligned} \tag{24}$$

And from equation (15) and inequalities (21) and (23), that

$$|K_{2s-1}(ze^{\pi i})| \leq \left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sqrt{e\left(\lceil 2s\rceil - \tfrac{1}{2}\right)}\right). \tag{25}$$

And similarly using (15), (17), and (19) we get that

$$|K_{2s-1}(ze^{-\pi i})| \leq \left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sqrt{e\left(\lceil 2s\rceil - \tfrac{1}{2}\right)}. \tag{26}$$

Thus we have that

$$\begin{aligned} |I_{2s-1}(z)| \;&\leq \frac{1}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sqrt{e\left(\lceil 2s\rceil - \tfrac{1}{2}\right)}\right) \\ &\quad + \frac{1}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sqrt{e\left(\lceil 2s\rceil - \tfrac{1}{2}\right)}\right) \\ &\quad + \frac{2}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z} \sum_{m=0}^{\lceil 2s-1\rceil} |a_m(2s-1)| \\ &\leq \frac{2}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sqrt{e\left(\lceil 2s\rceil - \tfrac{1}{2}\right)}\right) \\ &\quad + \frac{2}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z} \sum_{m=0}^{\lceil 2s-1\rceil} |a_m(2s-1)| \\ &\leq \left(\frac{2}{\pi z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil 2s-2\rceil} |a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right| \sqrt{e\left(\lceil 2s\rceil - \tfrac{1}{2}\right)} + \sum_{m=0}^{\lceil 2s-1\rceil} |a_m(2s-1)|\right) \end{aligned} \tag{27}$$

for $z \geq 1$. Recall the formula for the coefficient $b(n,\nu;s)$ for $n > 0$ is given by

$$b(n,\nu;s) = \sum_{c=1}^{\infty} \frac{S(n,-\nu;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right).$$

Utilizing the bound from (9) and splitting up the sum we get that

$$\begin{aligned}
|b(n,\nu;s)| &\leq \sum_{c=1}^{\infty} \frac{|S(n,-\nu;c)|}{c}\left| I_{2s-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right)\right| \\
&\leq \sum_{c=1}^{\infty} \frac{\tau(c)n^{1/2}}{c^{(1/2)}}\left| I_{2s-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right)\right| \\
&= n^{1/2}\left(\sum_{c=1}^{\lfloor 4\pi\sqrt{n\nu}\rfloor} \frac{\tau(c)}{c^{(1/2)}}\left| I_{2s-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right)\right| + \sum_{c>\lfloor 4\pi\sqrt{n\nu}\rfloor} \frac{\tau(c)}{c^{(1/2)}}\left| I_{2s-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right)\right|\right).
\end{aligned}$$

Plugging in the values of (13) and (27) and applying (10) and (11) yields

$$\begin{aligned}
|b(n,\nu;s)| \leq\; & n^{1/2}\left(\left(2\pi^2\sqrt{n\nu}\right)^{-1/2}\sum_{c=1}^{\lfloor 4\pi\sqrt{n\nu}\rfloor}\tau(c)e^{\frac{4\pi\sqrt{n\nu}}{c}}\right) \\
& \times\left(\sum_{m=0}^{\lceil 2s-1\rceil}|a_m(2s-1)| + \sum_{m=0}^{\lceil 2s-2\rceil}|a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right|\sqrt{e\left(\lceil 2s\rceil-\tfrac{1}{2}\right)}\right) \\
& + n^{1/2}\left(\left(2\pi\sqrt{n\nu}\right)^{2s-1}\sum_{k=0}^{\infty}\frac{1}{4^k k!\,|\Gamma(2s+k)|}\sum_{c>\lfloor 4\pi\sqrt{n\nu}\rfloor}\frac{\tau(c)}{c^{2s-\frac{1}{2}}}\right) \\
\leq\; & n^{1/2}\left((32\pi n\nu)^{1/2}C(1/2)e^{4\pi\sqrt{n\nu}}\right) \\
& \times\left(\sum_{m=0}^{\lceil 2s-1\rceil}|a_m(2s-1)| + \sum_{m=0}^{\lceil 2s-2\rceil}|a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right|\sqrt{e\left(\lceil 2s\rceil-\tfrac{1}{2}\right)}\right) \\
& + n^{1/2}\left(n^{s-\frac{1}{2}}(2\pi\sqrt{\nu})^{2s-1}\sum_{k=0}^{\infty}\frac{1}{4^k k!\,|\Gamma(2s+k)|}\zeta^2\left(2s-\tfrac{1}{2}\right)\right) \\
\leq\; & \left((32\pi\nu)^{1/2}C(1/2)\left(\sum_{m=0}^{\lceil 2s-1\rceil}|a_m(2s-1)| + \sum_{m=0}^{\lceil 2s-2\rceil}|a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right|\sqrt{e\left(\lceil 2s\rceil-\tfrac{1}{2}\right)}\right)\right. \\
& \left. +(4\pi^2\nu)^{s-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{1}{4^k k!\,|\Gamma(2s+k)|}\zeta^2\left(2s-\tfrac{1}{2}\right)\right)e^{4\pi\sqrt{n\nu}}n^s.
\end{aligned}$$

Thus if we define

$$\begin{aligned}
C_2(s,\nu) :=& (32\pi\nu)^{1/2}C(1/2)\left(\sum_{m=0}^{\lceil 2s-1\rceil}|a_m(2s-1)| + \sum_{m=0}^{\lceil 2s-2\rceil}|a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right|\sqrt{e\left(\lceil 2s\rceil-\tfrac{1}{2}\right)}\right) \\
& + (4\pi^2\nu)^{s-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{1}{4^k k!\,|\Gamma(2s+k)|}\zeta^2\left(2s-\tfrac{1}{2}\right) \qquad (28)
\end{aligned}$$

then

$$|b(n,\nu;s)| \leq C_2(s,\nu)e^{4\pi\sqrt{n\nu}}n^s$$

for $n>0$, as stated in Proposition 1. □

3. Proof of Theorem 1

Proof. Inserting the Fourier expansion of $F_{s,\nu}(z)$ into the definition of the trace, we get

$$\mathrm{Tr}_d(F_{s,\nu}) = \sum_{Q\in S_d} 2\pi\nu^{s-\frac{1}{2}}(\mathrm{Im}(\tau_Q))^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi\nu\mathrm{Im}(\tau_Q))e(-\nu\mathrm{Re}(\tau_Q)) + E(s,\nu,d)$$

where

$$E(s,\nu,d) := E_1(s,\nu,d) + E_2(s,\nu,d)$$

with

$$E_1(s,\nu,d) := \sum_{Q\in S_d} \frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)}(\mathrm{Im}(\tau_Q))^{1-s}$$

$$E_2(s,\nu,d) := \sum_{Q\in S_d} 4\pi\nu^{s-\frac{1}{2}} \sum_{n\neq 0} b(n,\nu;s)(\mathrm{Im}(\tau_Q))^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi\,|n|\,\mathrm{Im}(\tau_Q))e(n\mathrm{Re}(\tau_Q))$$

First we will bound $E_1(s,\nu,d)$. Since $s \geq 1$, we have $1-s \leq 0$. Also, as mentioned in the introduction, all of the τ_Q that we are summing over lie in the standard fundamental domain $\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})$. From the geometry of $\mathcal{F}$, it can be seen that any point lying in $\mathcal{F}$ has imaginary part greater than or equal to $\frac{\sqrt{3}}{2}$. Thus

$$\mathrm{Im}(\tau_Q) \geq \frac{\sqrt{3}}{2}$$

so

$$\begin{aligned}
|E_1(s,\nu,d)| &\leq \sum_{Q\in S_d}\left|\frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)}(\mathrm{Im}(\tau_Q))^{1-s}\right| \\
&\leq \sum_{Q\in S_d}\left|\frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)}\left(\frac{\sqrt{3}}{2}\right)^{1-s}\right| \\
&\leq \sum_{Q\in S_d}\left|\frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)}\right| \\
&= \sum_{Q\in S_d}\frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)} \\
&= \frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)}h(d). \qquad (29)
\end{aligned}$$

We will now bound $E_2(s,\nu,d)$. Let

$$E_3(s,\nu,d) := 4\pi\nu^{s-\frac{1}{2}}\sum_{Q\in S_d}\sum_{n<0} b(n,\nu;s)(\mathrm{Im}(\tau_Q))^{1/2}K_{s-\frac{1}{2}}(2\pi\,|n|\,\mathrm{Im}(\tau_Q))e(n\mathrm{Re}(\tau_Q)), \qquad (30)$$

$$E_4(s,\nu,d) := 4\pi\nu^{s-\frac{1}{2}}\sum_{Q\in S_d}\sum_{n>0} b(n,\nu;s)(\mathrm{Im}(\tau_Q))^{1/2}K_{s-\frac{1}{2}}(2\pi\,|n|\,\mathrm{Im}(\tau_Q))e(n\mathrm{Re}(\tau_Q)), \qquad (31)$$

so that

$$E_2(s,\nu,d) = E_3(s,\nu,d) + E_4(s,\nu,d).$$

In order to bound these error terms, we first bound $K_{s-\frac{1}{2}}(2\pi|n|\operatorname{Im}(\tau_Q))$. From [5, equation A.4] we get the asymptotic formula

$$K_{s-\frac{1}{2}}(2\pi|n|\operatorname{Im}(\tau_Q)) = \left(\frac{1}{4|n|\operatorname{Im}(\tau_Q)}\right)^{1/2} e^{-2\pi|n|\operatorname{Im}(\tau_Q)} \times \left(\sum_{m=0}^{\lceil s-1\rceil} \frac{a_m(s-\frac{1}{2})}{(2\pi|n|\operatorname{Im}(\tau_Q))^m} + R^{(K)}_{\lceil s\rceil}\left(2\pi|n|\operatorname{Im}(\tau_Q), s-\frac{1}{2}\right)\right). \tag{32}$$

Recall that $\operatorname{Im}(\tau_Q) \geq \frac{\sqrt{3}}{2}$. Using this and a similar method as that used to reduce inequality (17) we get that

$$\begin{aligned}\left|R^{(K)}_{\lceil s\rceil}\left(2\pi|n|\operatorname{Im}(\tau_Q), s-\frac{1}{2}\right)\right| &\leq \frac{\left|a_{\lceil s\rceil}\left(s-\frac{1}{2}\right)\right|}{|2\pi|n|\operatorname{Im}(\tau_Q)|^{\lceil s\rceil}} \sup_{r\geq 1}\left|\Lambda_{\lceil s\rceil+\max(0,1-s)}(4\pi|n|\operatorname{Im}(\tau_Q)r)\right| \\ &\leq \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right|.\end{aligned} \tag{33}$$

Then combining (32) and (33),

$$\begin{aligned}\left|K_{s-\frac{1}{2}}(2\pi|n|\operatorname{Im}(\tau_Q))\right| &\leq \left(\frac{1}{4|n|\operatorname{Im}(\tau_Q)}\right)^{1/2} e^{-2\pi|n|\operatorname{Im}(\tau_Q)} \\ &\quad\times \left(\sum_{m=0}^{\lceil s-1\rceil}\left|\frac{a_m\left(s-\frac{1}{2}\right)}{(2\pi|n|\operatorname{Im}(\tau_Q))^m}\right| + \left|R^{(K)}_{\lceil s\rceil}\left(2\pi|n|\operatorname{Im}(\tau_Q), s-\frac{1}{2}\right)\right|\right) \\ &\leq \left(\frac{1}{4|n|\operatorname{Im}(\tau_Q)}\right)^{1/2} e^{-2\pi|n|\operatorname{Im}(\tau_Q)}\left(\sum_{m=0}^{\lceil s-1\rceil}\left|a_m(s-\tfrac{1}{2})\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right|\right) \\ &= \left(\frac{1}{4|n|\operatorname{Im}(\tau_Q)}\right)^{1/2} e^{-2\pi|n|\operatorname{Im}(\tau_Q)}\left(\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|\right).\end{aligned} \tag{34}$$

We now bound $|E_3(s,\nu,d)|$. Recall from Proposition 1 that

$$|b(n,\nu;s)| \leq C_1(s,\nu)\,|n|^s$$

for $n<0$, where $C_1(s,\nu)$ is defined in equation (12). Combining this bound with equation (30) and inequality (34),

$$\begin{aligned}|E_3(s,\nu,d)| &\leq 2\pi\nu^{s-\frac{1}{2}}C_1(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|\sum_{Q\in S_d}\sum_{n<0}|n|^{s-\frac{1}{2}}e^{-2\pi|n|\operatorname{Im}(\tau_Q)}\left|e(n\operatorname{Re}(\tau_Q))\right| \\ &= 2\pi\nu^{s-\frac{1}{2}}C_1(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|\sum_{n<0}|n|^{s-\frac{1}{2}}\sum_{Q\in S_d}e^{-2\pi|n|\operatorname{Im}(\tau_Q)}.\end{aligned} \tag{35}$$

As observed earlier, $\operatorname{Im}(\tau_Q) \geq \frac{\sqrt{3}}{2}$, so

$$-2\pi n\operatorname{Im}(\tau_Q) \leq -\pi n\sqrt{3},$$

and thus

$$\begin{aligned}\sum_{Q\in S_d} e^{-2\pi|n|\operatorname{Im}(\tau_Q)} &\leq \sum_{Q\in S_d} e^{-\pi|n|\sqrt{3}} \\ &\leq h(d)e^{-\pi|n|\sqrt{3}}.\end{aligned} \tag{36}$$

Combining inequalities (35) and (36), we get

$$\begin{aligned}|E_3(s,\nu,d)| &\leq 2\pi\nu^{s-\frac{1}{2}}C_1(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|h(d)\sum_{n<0}|n|^{s-\frac{1}{2}}\,e^{-\pi|n|\sqrt{3}}\\ &\leq 2\pi\nu^{s-\frac{1}{2}}C_1(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|h(d)\int_0^\infty x^{s-\frac{1}{2}}e^{-\pi x\sqrt{3}}dx\\ &= 2\pi\nu^{s-\frac{1}{2}}C_1(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|h(d)\left(\frac{1}{\pi\sqrt{3}}\right)^{s+\frac{1}{2}}\Gamma\left(s+\frac{1}{2}\right)\\ &\leq 2\pi\nu^{s-\frac{1}{2}}h(d)C_1(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|\Gamma\left(s+\frac{1}{2}\right).\end{aligned}$$

We will now bound $|E_4(s,\nu,d)|$. From Proposition 1 we have that

$$|b(n,\nu;s)| \leq C_2(s,\nu)e^{4\pi\sqrt{n\nu}}n^s$$

for $n>0$, where $C_2(s,\nu)$ is defined in equation (28). Utilizing (31) and (34),

$$\begin{aligned}|E_4(s,\nu,d)| &\leq 2\pi\nu^{s-\frac{1}{2}}C_2(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|\sum_{n>0}\sum_{Q\in S_d}e^{4\pi\sqrt{n\nu}}n^{s-\frac{1}{2}}e^{-2\pi n\mathrm{Im}(\tau_Q)}\left|e(n\mathrm{Re}(\tau_Q))\right|\\ &\leq 2\pi\nu^{s-\frac{1}{2}}C_2(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|\sum_{n>0}e^{4\pi\sqrt{n\nu}}n^{s-\frac{1}{2}}\sum_{Q\in S_d}e^{-2\pi n\mathrm{Im}(\tau_Q)}.\end{aligned}$$

Similar reasoning as used to achieve (36) gives us that

$$\sum_{Q\in S_d}e^{-2\pi n\mathrm{Im}(\tau_Q)} \leq h(d)e^{-\pi n\sqrt{3}}.$$

And thus we get

$$|E_4(s,\nu,d)| \leq 2\pi\nu^{s-\frac{1}{2}}C_2(s,\nu)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|h(d)\sum_{n>0}n^{s-\frac{1}{2}}\exp(4\pi\sqrt{n\nu}-\pi n\sqrt{3}). \tag{37}$$

Observe that

$$4\pi\sqrt{n\nu}-\pi n\sqrt{3} = -n\left(\pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{n}}\right),$$

and note that when $n > \lfloor\frac{16\nu}{3}\rfloor$ we have

$$\pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{n}} > 0.$$

Let $n \geq \lceil\frac{16\nu}{3}\rceil$. Then

$$\pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{n}} \geq \pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{\lceil\frac{16\nu}{3}\rceil}}$$

and thus

$$-n\left(\pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{n}}\right) \leq -n\left(\pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{\lceil\frac{16\nu}{3}\rceil}}\right).$$

Let

$$\Sigma_1 := \sum_{n>0} n^{s-\frac{1}{2}} \exp(4\pi\sqrt{n\nu} - \pi n\sqrt{3}).$$

Observe

$$\begin{aligned}\Sigma_1 &= \sum_{n=1}^{\lfloor\frac{16\nu}{3}\rfloor} n^{s-\frac{1}{2}} \exp(4\pi\sqrt{n\nu} - \pi n\sqrt{3}) + \sum_{n=\lceil\frac{16\nu}{3}\rceil}^{\infty} n^{s-\frac{1}{2}} \exp\left(-n\left(\pi\sqrt{3} - \frac{4\pi\sqrt{\nu}}{\sqrt{n}}\right)\right)\\ &\le \sum_{n=1}^{\lfloor\frac{16\nu}{3}\rfloor} n^{s-\frac{1}{2}} \exp(4\pi\sqrt{n\nu} - \pi n\sqrt{3}) + \sum_{n=\lceil\frac{16\nu}{3}\rceil}^{\infty} n^{s-\frac{1}{2}} \exp\left(-n\left(\pi\sqrt{3} - \frac{4\pi\sqrt{\nu}}{\sqrt{\lceil\frac{16\nu}{3}\rceil}}\right)\right).\end{aligned}$$

Let

$$\Sigma_2 := \sum_{n=1}^{\lfloor\frac{16\nu}{3}\rfloor} n^{s-\frac{1}{2}} \exp(4\pi\sqrt{n\nu} - \pi n\sqrt{3}).$$

and

$$\Sigma_3 := \sum_{n=\lceil\frac{16\nu}{3}\rceil}^{\infty} n^{s-\frac{1}{2}} \exp\left(-n\left(\pi\sqrt{3} - \frac{4\pi\sqrt{\nu}}{\sqrt{\lceil\frac{16\nu}{3}\rceil}}\right)\right).$$

First note that

$$\Sigma_2 \le \int_1^{\lfloor\frac{16\nu}{3}\rfloor} x^{s-\frac{1}{2}} \exp(4\pi\sqrt{x\nu} - \pi n\sqrt{3})dx.$$

Then let

$$g(\nu) := \pi\sqrt{3} - \frac{4\pi\sqrt{\nu}}{\sqrt{\lceil\frac{16\nu}{3}\rceil}}.$$

Note that $g(\nu) > 0$, so $\exp(-g(\nu)x) < \exp(-x)$ for all $x \in [\lceil\frac{16\nu}{3}\rceil, \infty)$. Thus

$$\begin{aligned}\Sigma_3 &\le \int_{\lceil\frac{16\nu}{3}\rceil}^{\infty} x^{s-\frac{1}{2}} \exp(-g(\nu)x)dx\\ &\le \int_{\lceil\frac{16\nu}{3}\rceil}^{\infty} x^{s-\frac{1}{2}} \exp(-x)dx\\ &= \Gamma\left(s+\frac{1}{2}, \left\lceil\frac{16\nu}{3}\right\rceil\right).\end{aligned}$$

Thus from (37),

$$\begin{aligned}|E_4(s,\nu,d)| &\le 2\pi\nu^{s-\frac{1}{2}} C_2(s,\nu)h(d)\sum_{m=0}^{\lceil s\rceil} \left|a_m(s-\tfrac{1}{2})\right| \Sigma_1\\ &\le 2\pi\nu^{s-\frac{1}{2}} C_2(s,\nu)h(d)\sum_{m=0}^{\lceil s\rceil} \left|a_m(s-\tfrac{1}{2})\right| (\Sigma_2+\Sigma_3)\\ &\le 2\pi\nu^{s-\frac{1}{2}} C_2(s,\nu)h(d)\sum_{m=0}^{\lceil s\rceil} \left|a_m(s-\tfrac{1}{2})\right| \left(\Gamma\left(s+\frac{1}{2}, \left\lceil\frac{16\nu}{3}\right\rceil\right) + \int_1^{\lfloor\frac{16\nu}{3}\rfloor} x^{s-\frac{1}{2}} \exp(4\pi\sqrt{x\nu} - \pi n\sqrt{3})dx\right).\end{aligned}$$

In summary, by combining our bounds for $E_1(s,\nu,d)$, $E_3(s,\nu,d)$, and $E_4(s,\nu,d)$ we get that

$$
\begin{aligned}
|E(s,\nu,d)| \leq & |E_1(s,\nu,d)| + |E_3(s,\nu,d)| + |E_4(s,\nu,d)| \qquad (38)\\
\leq & \frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)} h(d)\\
& + 2\pi\nu^{s-\frac{1}{2}} C_1(s,\nu) \sum_{m=0}^{\lceil s \rceil} \left|a_m(s-\tfrac{1}{2})\right| \Gamma\left(s+\frac{1}{2}\right) h(d)\\
& + 2\pi\nu^{s-\frac{1}{2}} C_2(s,\nu) \sum_{m=0}^{\lceil s \rceil} \left|a_m(s-\tfrac{1}{2})\right| \left(\Gamma\left(s+\frac{1}{2}, \left\lceil \frac{16\nu}{3} \right\rceil\right)\right.\\
& \left. + \int_1^{\lfloor \frac{16\nu}{3} \rfloor} x^{s-\frac{1}{2}} \exp(4\pi\sqrt{x\nu} - \pi n\sqrt{3}) dx \right) h(d)\\
= & \left(\frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)} \right.\\
& + \left((32\pi\nu)^{1/2} C(1/2) \sum_{k=0}^{2N+1} |a_k(2s-1)|) + (4\pi^2\nu)^{s-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{4^k k! \, |\Gamma(2s+k)|} \zeta^2\left(2s-\tfrac{1}{2}\right) \right)\\
& \times \sum_{m=0}^{\lceil s \rceil} \left|a_m(s-\tfrac{1}{2})\right| \Gamma\left(s+\frac{1}{2}\right) 2\pi\nu^{s-\frac{1}{2}}\\
& + \left((32\pi\nu)^{1/2} C(1/2) \left(\sum_{m=0}^{\lceil 2s-1 \rceil} |a_m(2s-1)| + \sum_{m=0}^{\lceil 2s-2 \rceil} |a_m(2s-1)| + \left|a_{\lceil 2s-1 \rceil}(2s-1)\right| \sqrt{e\left(\lceil 2s \rceil - \tfrac{1}{2}\right)} \right) \right.\\
& \left. + (4\pi^2\nu)^{s-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{4^k k! \, |\Gamma(2s+k)|} \zeta^2\left(2s-\tfrac{1}{2}\right) \right) \sum_{m=0}^{\lceil s \rceil} \left|a_m(s-\tfrac{1}{2})\right| \left(\Gamma\left(s+\frac{1}{2}, \left\lceil \frac{16\nu}{3} \right\rceil\right)\right.\\
& \left.\left. + \int_1^{\lfloor \frac{16\nu}{3} \rfloor} x^{s-\frac{1}{2}} \exp(4\pi\sqrt{x\nu} - \pi n\sqrt{3}) dx \right) 2\pi\nu^{s-\frac{1}{2}} \right) h(d)
\end{aligned}
$$

Thus if we let

$$C(s,\nu) := \frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)} \tag{39}$$

$$+\left((32\pi\nu)^{1/2}C(1/2)\sum_{k=0}^{2N+1}|a_k(2s-1)|) + (4\pi^2\nu)^{s-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{1}{4^k k!\,|\Gamma(2s+k)|}\zeta^2\left(2s-\tfrac{1}{2}\right)\right)$$

$$\times\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|\Gamma\left(s+\frac{1}{2}\right)2\pi\nu^{s-\frac{1}{2}}$$

$$+\left((32\pi\nu)^{1/2}C(1/2)\left(\sum_{m=0}^{\lceil 2s-1\rceil}|a_m(2s-1)| + \sum_{m=0}^{\lceil 2s-2\rceil}|a_m(2s-1)| + \left|a_{\lceil 2s-1\rceil}(2s-1)\right|\sqrt{e\left(\lceil 2s\rceil-\tfrac{1}{2}\right)}\right)\right.$$

$$\left.+(4\pi^2\nu)^{s-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{1}{4^k k!\,|\Gamma(2s+k)|}\zeta^2\left(2s-\tfrac{1}{2}\right)\right)\sum_{m=0}^{\lceil s\rceil}\left|a_m(s-\tfrac{1}{2})\right|\left(\Gamma\left(s+\frac{1}{2},\left\lceil\frac{16\nu}{3}\right\rceil\right)\right)$$

$$\left.+\int_1^{\lfloor\frac{16\nu}{3}\rfloor}x^{s-\frac{1}{2}}\exp(4\pi\sqrt{x\nu}-\pi n\sqrt{3})dx\right)2\pi\nu^{s-\frac{1}{2}}$$

then

$$\mathrm{Tr}_d(F_{s,\nu}) = \sum_{Q\in S_d}2\pi\nu^{s-\frac{1}{2}}(\mathrm{Im}(\tau_Q))^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi\nu\mathrm{Im}(\tau_Q))e(-\nu\mathrm{Re}(\tau_Q)) + E(s,\nu,d) \tag{40}$$

where $E(s,\nu,d)$ satisfies the bound

$$|E(s,\nu,d)| \leq C(s,\nu)h(d),$$

as stated in Theorem 1.

We now prove our upper bound for $\mathrm{Tr}_d(F_{s,\nu}(z))$. All that remains is to bound the term

$$\Sigma_4 := \sum_{Q\in S_d}2\pi\nu^{s-\frac{1}{2}}(\mathrm{Im}(\tau_Q))^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi\nu\mathrm{Im}(\tau_Q))e(-\nu\mathrm{Re}(\tau_Q)).$$

First,

$$\begin{aligned}|\Sigma_4| &\leq \sum_{Q\in S_d}\left|2\pi\nu^{s-\frac{1}{2}}(\mathrm{Im}(\tau_Q))^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi\nu\mathrm{Im}(\tau_Q))e(-\nu\mathrm{Re}(\tau_Q))\right| \\ &= 2\pi\nu^{s-\frac{1}{2}}\sum_{Q\in S_d}(\mathrm{Im}(\tau_Q))^{\frac{1}{2}}\left|I_{s-\frac{1}{2}}(2\pi\nu\mathrm{Im}(\tau_Q))\right|\end{aligned} \tag{41}$$

To bound Σ_4, we first bound $I_{s-\frac{1}{2}}(z)$ for any real number $z\geq 1$. From [5, equation A.4] we get the identity

$$I_{s-\frac{1}{2}}(z) = \mp\frac{i}{\pi}K_{s-\frac{1}{2}}(ze^{\mp\pi i}) \pm \frac{i}{\pi}e^{\pm\pi i(s-\frac{1}{2})}K_{s-\frac{1}{2}}(z) \tag{42}$$

and the asymptotic formulas

$$K_{s-\frac{1}{2}}(ze^{\pm\pi i}) = \pm i\left(\frac{\pi}{2z}\right)^{(1/2)}e^z\left(\sum_{m=0}^{M-1}(-1)^m\frac{a_m\left(s-\frac{1}{2}\right)}{z^m} + R_M^{(K)}\left(ze^{\mp\pi i},s-\tfrac{1}{2}\right)\right) \tag{43}$$

and

$$K_{s-\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{(1/2)}e^{-z}\left(\sum_{m=0}^{M-1}\frac{a_m\left(s-\frac{1}{2}\right)}{z^m} + R_M^{(K)}\left(z,s-\tfrac{1}{2}\right)\right) \tag{44}$$

where $M := \lceil s \rceil$. From (42) we can write

$$|I_{s-\frac{1}{2}}(z)| \leq \frac{1}{\pi}|K_{s-\frac{1}{2}}(ze^{-\pi i})| + \frac{1}{\pi}|K_{s-\frac{1}{2}}(ze^{\pi i})| + \frac{2}{\pi}|K_{s-\frac{1}{2}}(z)|.$$

Now [5, equation 1.26] from Neme's Theorem 1.5 provides that for $|\arg w| < \frac{3\pi}{2}$ we get that

$$\left|R_M^{(K)}(w,\nu)\right| \leq \frac{|a_M(\nu)|}{|w|^M} \sup_{r\geq 1} \left|\Lambda_{M+\max(0,\frac{1}{2}-\nu)}(2wr)\right| \tag{45}$$

giving us that

$$\left|R_{\lceil s\rceil}^{(K)}\left(z, s-\tfrac{1}{2}\right)\right| \leq \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sup_{r\geq 1}\left|\Lambda_{\lceil s\rceil}(2zr)\right|$$

and

$$\left|R_{\lceil s\rceil}^{(K)}\left(ze^{\pi i}, s-\tfrac{1}{2}\right)\right| \leq \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sup_{r\geq 1}\left|\Lambda_{\lceil s\rceil}(2ze^{\pi i}r)\right|.$$

And then [5, Proposition B.1] tells us that

$$\left|\Lambda_{\lceil s\rceil}(2zr)\right| \leq 1, \tag{46}$$

and

$$\left|\Lambda_{\lceil s\rceil}(2ze^{\pm\pi i}r)\right| \leq \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)}. \tag{47}$$

Thus from (46) and the fact that $z \geq 1$, we get that

$$\left|R_{\lceil s\rceil}^{(K)}\left(z, s-\tfrac{1}{2}\right)\right| \leq \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \tag{48}$$

and similarly, using (47)

$$\left|R_{\lceil s\rceil}^{(K)}\left(ze^{\pi i}, s-\tfrac{1}{2}\right)\right| \leq \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)}. \tag{49}$$

Also observe that as $z \geq 1$

$$\sum_{m=0}^{\lceil s\rceil-1}\left|\frac{a_m\left(s-\frac{1}{2}\right)}{z^m}\right| \leq \sum_{m=0}^{\lceil s\rceil-1}\left|a_m\left(s-\tfrac{1}{2}\right)\right| \tag{50}$$

and

$$\sum_{m=0}^{\lceil s\rceil-1}\left|(-1)^m\frac{a_m\left(s-\frac{1}{2}\right)}{z^m}\right| \leq \sum_{m=0}^{\lceil s\rceil-1}\left|a_m\left(s-\tfrac{1}{2}\right)\right|. \tag{51}$$

Using equation (44) and inequalities (48) and (50), we see that

$$\begin{aligned}|K_{s-\frac{1}{2}}(z)| &\leq \left|\left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z}\right| \left(\sum_{m=0}^{\lceil s\rceil-1}\left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right|\right) \\ &= \left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z}\sum_{m=0}^{\lceil s\rceil}\left|a_m\left(s-\tfrac{1}{2}\right)\right|.\end{aligned} \tag{52}$$

And from equation (43) and inequalities (49) and (51), that

$$|K_{s-\frac{1}{2}}(ze^{\pi i})| \leq \left(\frac{\pi}{2z}\right)^{(1/2)} e^{z}\left(\sum_{m=0}^{\lceil s\rceil-1}\left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right|\sqrt{e\left(\lceil s\rceil+\tfrac{1}{2}\right)}\right). \tag{53}$$

And similarly using (43), (45), and (47) we get that

$$|K_{s-\frac{1}{2}}(ze^{-\pi i})| \leq \left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil s\rceil -1} \left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)}\right). \tag{54}$$

Thus we have that

$$\begin{aligned}
|I_{s-\frac{1}{2}}(z)| \; &\leq \frac{1}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil s\rceil -1} \left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)}\right) \qquad (55)\\
&\quad + \frac{1}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil s\rceil -1} \left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)}\right)\\
&\quad + \frac{2}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z} \sum_{m=0}^{\lceil s\rceil} \left|a_m\left(s-\tfrac{1}{2}\right)\right|\\
&\leq \frac{2}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil s\rceil -1} \left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)}\right)\\
&\quad + \frac{2}{\pi}\left(\frac{\pi}{2z}\right)^{(1/2)} e^{-z} \sum_{m=0}^{\lceil s\rceil} \left|a_m\left(s-\tfrac{1}{2}\right)\right|\\
&\leq \left(\frac{2}{\pi z}\right)^{(1/2)} e^{z} \left(\sum_{m=0}^{\lceil s\rceil -1} \left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)} + \sum_{m=0}^{\lceil s\rceil} \left|a_m\left(s-\tfrac{1}{2}\right)\right|\right).
\end{aligned}$$

Inserting 55 into **??**, we get

$$|\Sigma_4| \leq \pi\nu^{s-1} \sum_{Q\in S_d} e^{2\pi\nu \mathrm{Im}(\tau_Q)} \left(\sum_{m=0}^{\lceil s\rceil -1} \left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right| \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)} + \sum_{m=0}^{\lceil s\rceil} \left|a_m\left(s-\tfrac{1}{2}\right)\right|\right)$$

Now, since

$$\mathrm{Im}(\tau_Q) = \frac{\sqrt{|d|}}{2a_Q}$$

we have

$$\mathrm{Im}(\tau_Q) \leq \frac{\sqrt{|d|}}{2}$$

Thus

$$|\Sigma_4| \leq \pi\nu^{s-1} \left(\sum_{m=0}^{\lceil s\rceil -1} \left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\frac{1}{2}\right)\right| \sqrt{e\left(\lceil s\rceil + \tfrac{1}{2}\right)} + \sum_{m=0}^{\lceil s\rceil} \left|a_m\left(s-\tfrac{1}{2}\right)\right|\right) h(d) e^{\pi\nu\sqrt{|d|}}$$

Putting this together with **??** and **??**, we get that

$$\begin{aligned}|\mathrm{Tr}_d(F_{s,\nu}(z))| &\leq |\Sigma_4| + |E(s,\nu,d)| \\ &\leq \pi\nu^{s-1}\left(\sum_{m=0}^{\lceil s\rceil -1}\left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right|\sqrt{e\left(\lceil s\rceil+\tfrac{1}{2}\right)} + \sum_{m=0}^{\lceil s\rceil}\left|a_m\left(s-\tfrac{1}{2}\right)\right|\right)h(d)e^{\pi\nu\sqrt{|d|}} \\ &\quad + C(s,\nu)h(d) \\ &\leq \left(\pi\nu^{s-1}\left(\sum_{m=0}^{\lceil s\rceil -1}\left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right|\sqrt{e\left(\lceil s\rceil+\tfrac{1}{2}\right)} + \sum_{m=0}^{\lceil s\rceil}\left|a_m\left(s-\tfrac{1}{2}\right)\right|\right) + C(s,\nu)\right) \\ &\quad \times h(d)e^{\pi\nu\sqrt{|d|}}\end{aligned}$$

So if we let

$$C'(s,\nu) = \pi\nu^{s-1}\left(\sum_{m=0}^{\lceil s\rceil -1}\left|a_m\left(s-\tfrac{1}{2}\right)\right| + \left|a_{\lceil s\rceil}\left(s-\tfrac{1}{2}\right)\right|\sqrt{e\left(\lceil s\rceil+\tfrac{1}{2}\right)} + \sum_{m=0}^{\lceil s\rceil}\left|a_m\left(s-\tfrac{1}{2}\right)\right|\right) + C(s,\nu) \tag{56}$$

then we get

$$|\mathrm{Tr}_d(F_{s,\nu}(z))| \leq C'(s,\nu)h(d)e^{\pi\nu\sqrt{|d|}}$$

as stated in Theorem 1. □

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