An average of generalized Dedekind sums

Travis Dillon[1] & Stephanie Gaston[2]

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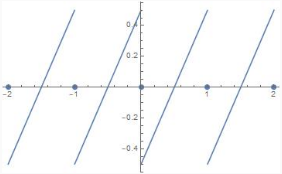
[1]Lawrence University
[2]California State University Dominguez Hills

Classical Dedekind Sum

Definition

$$B_1(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & \text{otherwise.} \end{cases}$$



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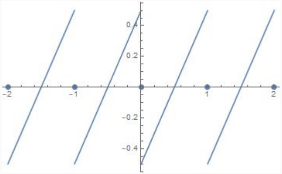

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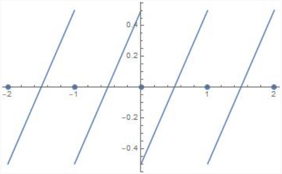

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. . . one of its many guises:

$$s(a,c) = \frac{1}{4c} \sum_{j \bmod c}{}' \cot\left(\frac{\pi j}{c}\right) \cot\left(\frac{\pi a j}{c}\right)$$

Dirichlet Characters

A **Dirichlet character** modulo q is a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ that has

1. period q
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if and only if $\gcd(n, q) > 1$
4. $\chi(1) = 1$

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n	0	1	2	3	4
$\chi(n)$	0	1	$-i$	i	-1

Primitive Characters I

The function

$$\chi_{0,m}(n) = \begin{cases} 1 & \text{if } \gcd(n,m) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

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n	0	1	2	3	4
$\psi(n)$	0	1	i	$-i$	-1

n	0	1	2	3	4	5	6	7	8	9
$\psi\chi_{0,2}(n)$	0	1	0	$-i$	0	0	0	i	0	-1

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A **primitive** character is not induced by any other character.

Primitive Characters II

n	0	1	2	3	4	5	6	7	8	9	10	11
$\psi(n)$	0	1	0	0	0	-1	0	1	0	0	0	-1

Primitive Characters II

n	0	1	2	3	4	5	6	7	8	9	10	11
$\psi(n)$	0	1	0	0	0	−1	0	1	0	0	0	−1

Primitive Characters II

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$\psi(n)$	0	1	0	0	0	−1	0	1	0	0	0	−1
$\psi^{\star}(n)$	0	1	−1	0	1	−1	0	1	−1	0	1	−1

n	0	1	2
$\psi^{\star}(n)$	0	1	−1

The **Dirichlet L-function** associated with the character χ is

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

The L-function

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Dirichlet used $L(1,\chi)$ to study primes in arithmetic progressions

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Walum evaluated

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2.$$

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Theorem (Walum, 1982)

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^4 = \frac{\pi^4(p-1)}{p^2} \sum_{a \bmod p} |s(a,c)|^2.$$

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Rearranging, we have an average of Dedekind sums:

$$\sum_{a \bmod p} |s(a,p)|^2 = \frac{p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^4.$$

Generalized Dedekind Sum

Definition

Let χ_1 mod q_1 and χ_2 mod q_2 be non-trivial primitive Dirichlet characters. The **generalized Dedekind sum** is

$$S_{\chi_1,\chi_2}(a,c) = \sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_2}(j)\overline{\chi_1}(n)\, B_1\!\left(\frac{j}{c}\right) B_1\!\left(\frac{n}{q_1} + \frac{aj}{c}\right)$$

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$$S_{\chi_1,\chi_2}(a,c) = K \sum_{s \bmod c}{}' \sum_{r \bmod q_2}{}' \chi_1(s)\chi_2(r) \cot\left(\pi\left(\frac{r}{q_2} - \frac{as}{c}\right)\right) \cot\left(\frac{\pi s}{c}\right)$$

The Second Moment

Theorem (D. and G., 2019)

Let χ_1 and χ_2 be nontrivial primitive characters such that $\chi_1\chi_2(-1) = 1$, and let $q_1 q_2 \mid c$. Then

$$\sum_{\substack{a \bmod c \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^2 = \frac{\varphi(c)}{\pi^4} \sum_{\substack{\psi \bmod c \\ \psi\chi_1(-1)=-1}} |L(1, \overline{\psi}^\star \chi_1)|^2 |L(1, (\psi\chi_2)^\star)|^2 |g_{\chi_1,\chi_2}(\psi; c)|^2.$$

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$$g_{\chi_1,\chi_2}(\psi; c) = K(\psi) \sum_{\substack{d|c \\ d\equiv 0 \bmod q(\psi)}} \frac{\overline{\chi_2}(c/d)}{\varphi(d)} ((\overline{\psi\chi_2})^\star \mu * 1)(d)\ (\chi_1 * \mu\psi^\star)\left(\frac{d}{q(\psi)}\right)$$

Second Moment Bound

Theorem (D. and G., 2019)

Let χ_1 and χ_2 be nontrivial primitive characters modulo q_1 and q_2, respectively, such that $\chi_1\chi_2(-1) = 1$, and let $q_1q_2 \mid c$. For every $\varepsilon > 0$, there exist positive A_ε and B_ε such that

$$A_\varepsilon c^{2-\varepsilon} \leq \sum_{\substack{a \bmod c \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^2 \leq B_\varepsilon c^{2+\varepsilon}.$$

Corollary

*For all $c > 0$, $S_{\chi_1,\chi_2}(a,c)$ does **not** vanish.*

A Different View

$SL_2(\mathbb{Z})$

Definition

The special linear group $\mathbf{SL_2}(\mathbb{Z})$ is the set of 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $ad - bc = 1$.

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Definition

For $N \in \mathbb{N}^+$, the subgroup of $SL_2(\mathbb{Z})$ such that N divides c is denoted $\Gamma_0(N)$.

The Dedekind sum is a map from $\Gamma_0(q_1 q_2)$ to $\mathbb{C}$ by

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_1,\chi_2}(a, c).$$

A Map with Structure

Let $\chi(\gamma) = \chi(d)$. Then

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \chi_1\overline{\chi_2}(\gamma_1)S_{\chi_1,\chi_2}(\gamma_2).$$

If $\chi_1 = \chi_2$, then $\chi_1\overline{\chi_2}(\gamma_1) = 1$, so $S_{\chi_1,\chi_2}(\gamma)$ is a homomorphism.

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Corollary

The crossed homomorphism S_{χ_1,χ_2} is nontrivial. In fact, for each $c > 0$, there exists some $a \in \mathbb{Z}$ so that $S_{\chi_1,\chi_2}(a, c) \neq 0$.

Questions?

Bounds on the Second Moment

Overview

Recall that:

$$A_\varepsilon c^{2-\varepsilon} \leq \sum_{\substack{a \bmod c \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^2 \leq B_\varepsilon c^{2+\varepsilon}$$

Sketchy Outline: Upper bound

$$\sum_{\substack{a \bmod c \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^2 = \frac{\varphi(c)}{\pi^4} \sum_{\substack{\psi \bmod c \\ \psi\chi_1(-1)=-1}} |L(1,\overline{\psi}^\star\chi_1)|^2 |L(1,(\psi\chi_2)^\star)|^2 |g_{\chi_1,\chi_2}(\psi;c)|^2$$

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Bound the L-functions:

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- Bound by divisor function

Divisor Function

Definition

$d(n)$ is the number of positive divisors of n.

Example: The divisors of 12 are $\{1, 2, 3, 4, 6, 12\}$, so $d(12) = 6$.

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Property

If $\gcd(m, n) = 1$, then $d(mn) = d(m)d(n)$.

So look at $d(p^k)$ for primes p.

Divisor Bound

Want to show that $d(p^k) \leq K_\varepsilon p^{k\varepsilon}$, so consider

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Calculate: $d(p^k) = k + 1$.

$$\frac{k+1}{(p^\varepsilon)^k} \leq K_\varepsilon$$

Therefore $d(n) \leq K_\varepsilon n^\varepsilon$.

Sketchy Outline: Lower bound

$$\sum_{\substack{a \bmod c \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^2 \geq A_\varepsilon c^{2-\varepsilon}$$

Sketchy Outline: Lower bound

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Bound the L-functions:

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Bound g:

- Restrict the sum

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Bound g:

- Restrict the sum
- All the terms are 1!
- Clever counting

A lemma that counts I

Question

How many primitive characters modulo q are there?

Recall that a primitive character is **not** induced by a character of lower modulus.

Let $\varphi^\star(q)$ be the number of primitive characters modulo q.

Pick a prime . . .

Look at characters modulo p^n.

Idea: count the opposite.

Pick a prime . . .

Look at characters modulo p^n.

Idea: count the opposite.

A character is **not** primitive if it is induced by a character modulo p^{n-1}.

So we just need to find the number of characters modulo p^{n-1}.

A Dirichlet Digression

Definition

Let $n \in \mathbb{N}^+$. The set

$$(\mathbb{Z}/n\mathbb{Z})^* := \{a \in \mathbb{Z}/n\mathbb{Z} : \gcd(a, n) = 1\}$$

is a group under multiplication.

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We can also define a Dirichlet character χ mod q as a homomorphism $(\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$. (This means that $\chi(1) = 1$ and $\chi(mn) = \chi(m)\chi(n)$.)

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We can also define a Dirichlet character χ mod q as a homomorphism $(\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$. (This means that $\chi(1) = 1$ and $\chi(mn) = \chi(m)\chi(n)$.)

Then extend χ to $\mathbb{Z}$ by setting

$$\chi(n) = \begin{cases} \chi(n \bmod q) & \text{if } \gcd(n, q) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

A Dirichlet Digression

Fact

The number of characters modulo q is equal to the number of elements of $(\mathbb{Z}/q\mathbb{Z})^*$.

Definition

The number of positive integers less than q that are relatively prime to q is denoted $\varphi(q)$.

So there are $\varphi(p^{n-1})$ characters modulo p^{n-1}.

A lemma that counts II

Modulo p^n, there are

1. $\varphi(p^n)$ characters
2. $\varphi(p^{n-1})$ imprimitive characters
3. $\varphi(p^n) - \varphi(p^{n-1})$ primitive characters.

A lemma that counts II

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Proposition

$$\varphi^\star(p^n) = p^n - 2p^{n-1} + p^{n-2}.$$

Conclusion

Conclusion, being the Place in which we Recapitulate the High Points previously stated to you Fine Folk, and including a Small Sampling of the Exceedingly Excellent Problems related thereto

- S_{χ_1,χ_2} is a generalization of Dedekind sum
- $S_{\chi_1,\chi_2} : \Gamma_0(q_1 q_2) \to \mathbb{C}$
- Exact formula and bounds for second moment
- Proved that S_{χ_1,χ_2} is always a nontrivial map into $\mathbb{C}$.

Future work

Find formula for or asymptotics of higher moments

Thank You!

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