The image of the generalized Dedekind sum

Evelyne Knight, Carlos Matos, Amira Sefidi

Texas A&M University

June 2024

Outline

Introduction

Dirichlet Characters

Definition (Dirichlet Character)

A **Dirichlet Character**, $\chi \pmod q$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying

1. $\chi(n + ql) = \chi(n)$ for all $n, l \in \mathbb{Z}$ *(periodicity)*
2. $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$ *(multiplicativity)*
3. $\chi(n) = 0$ if and only if $\gcd(n, q) > 1$ *(coprimality)*

Dirichlet Characters

Definition (Dirichlet Character)

A **Dirichlet Character**, $\chi \pmod q$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying

1. $\chi(n + ql) = \chi(n)$ for all $n, l \in \mathbb{Z}$ *(periodicity)*
2. $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$ *(multiplicativity)*
3. $\chi(n) = 0$ if and only if $\gcd(n, q) > 1$ *(coprimality)*

Example. Here is a table of the dirichlet characters modulo 5.

χ, n	0	1	2	3	4
χ_0	0	1	1	1	1
χ_1	0	1	-1	-1	1
χ_2	0	1	i	$-i$	-1
χ_3	0	1	$-i$	i	-1

Dirichlet Characters-*Properties*

Here are a couple of useful properties and facts about **Dirichlet Characters**:

Dirichlet Characters-*Properties*

Here are a couple of useful properties and facts about **Dirichlet Characters**:

- Dirichlet characters modulo q form a *group* under multiplication. The **trivial character**, $\chi_{\mathbf{0}}$, is defined by $\chi_{\mathbf{0}}(n) = 1$ for all $n \in (\mathbb{Z}/q\mathbb{Z})^{\times}$ and $\chi_{\mathbf{0}} = 0$ otherwise.

Dirichlet Characters-*Properties*

Here are a couple of useful properties and facts about **Dirichlet Characters**:

- Dirichlet characters modulo q form a *group* under multiplication. The **trivial character**, $\chi_{\mathbf{0}}$, is defined by $\chi_{\mathbf{0}}(n) = 1$ for all $n \in (\mathbb{Z}/q\mathbb{Z})^{\times}$ and $\chi_{\mathbf{0}} = 0$ otherwise.
  - In fact, they are in a one-to-one correspondence with elements of $(\mathbb{Z}/\mathbf{q}\mathbb{Z})^{\times}$

Dirichlet Characters-*Properties*

Here are a couple of useful properties and facts about **Dirichlet Characters**:

- Dirichlet characters modulo q form a *group* under multiplication. The **trivial character**, $\chi_{\mathbf{0}}$, is defined by $\chi_{\mathbf{0}}(n) = 1$ for all $n \in (\mathbb{Z}/q\mathbb{Z})^{\times}$ and $\chi_{\mathbf{0}} = 0$ otherwise.
  - In fact, they are in a one-to-one correspondence with elements of $(\mathbb{Z}/\mathbf{q}\mathbb{Z})^{\times}$
- They have a natural connection with the *roots of unity*: if χ is of modulus q, then $\text{Im}(\chi) \subseteq \mathbb{Q}(\zeta_{\varphi(\mathbf{q})})$. Put in other words, $\chi(n)^k = 1$ for some $k \in \mathbb{Z}$.

Dirichlet Characters-*Properties*

Here are a couple of useful properties and facts about **Dirichlet Characters**:

- Dirichlet characters modulo q form a *group* under multiplication. The **trivial character**, $\chi_{\mathbf{0}}$, is defined by $\chi_{\mathbf{0}}(n) = 1$ for all $n \in (\mathbb{Z}/q\mathbb{Z})^\times$ and $\chi_{\mathbf{0}} = 0$ otherwise.
  - In fact, they are in a one-to-one correspondence with elements of $(\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times$
- They have a natural connection with the *roots of unity*: if χ is of modulus q, then $\text{Im}(\chi) \subseteq \mathbb{Q}(\zeta_{\varphi(\mathbf{q})})$. Put in other words, $\chi(n)^k = 1$ for some $k \in \mathbb{Z}$.

Modular Forms

Definition (Modular Forms)

A *modular form* of weight k for $\Gamma = SL_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that

1. For all $\gamma \in \Gamma$, $z \in \mathbb{H}$, we have
$$f\left(\frac{az+b}{cz+d}\right) = \epsilon(a,b,c,d)(cz+d)^k f(z).$$
2. The limit $y \to \infty$ of $f(x+iy)$ exists (and is not ∞)
3. $f(z)$ is complex analytic (i.e. complex differentiable for all $z \in \mathbb{H}$).

One of the most pivotal roles of modular forms was in the proof of Fermat's last theorem. They are also used in the partition function, and figuring out the densest way to pack spheres!

Dedekind η-function

The **Dedekind η function** is an example of a modular form which is used to study the following counting problem:

Definition (The partition function)

$\mathbf{p(n)}$ counts the number of ways to write n as a sum of positive integers in decreasing order.

Example.
$\mathbf{p(5)} = 7$, since $5 = 5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$.

The η-Function & the Partition Function

The *generating function* of $\mathbf{p(n)}$, $F(q)$, is related to the **Dedekind η function**, by

$$F(q) = \sum_{n=0}^{\infty} \mathbf{p(n)} q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1} = \frac{q^{1/24}}{\eta(z)}$$

Definition (Dedekind η - function)

For $z \in \mathbb{H}$, define

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

The Dedekind Eta function and the Dedekind Sum

The η function is a modular form of weight $1/2$ and satisfies the transformation law

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon(a,b,c,d)(cz+d)^{\frac{1}{2}}\eta(z),$$

where

$$\epsilon(a,b,c,d) := \begin{cases} e^{\frac{bi\pi}{12}}, & c = 0, d = 1 \\ e^{i\pi\left(\frac{a+d}{12c} - S(d,c) - \frac{1}{4}\right)}, & c > 0. \end{cases}$$

The Dedekind sum was first introduced to study the automorphy factor for the transformation of the Dedekind η function.
It has also appeared outside of number theory, for example in the enumeration of lattice points in tetrahedra.

Classic Dedekind Sum

Definition (Dedekind Sum)

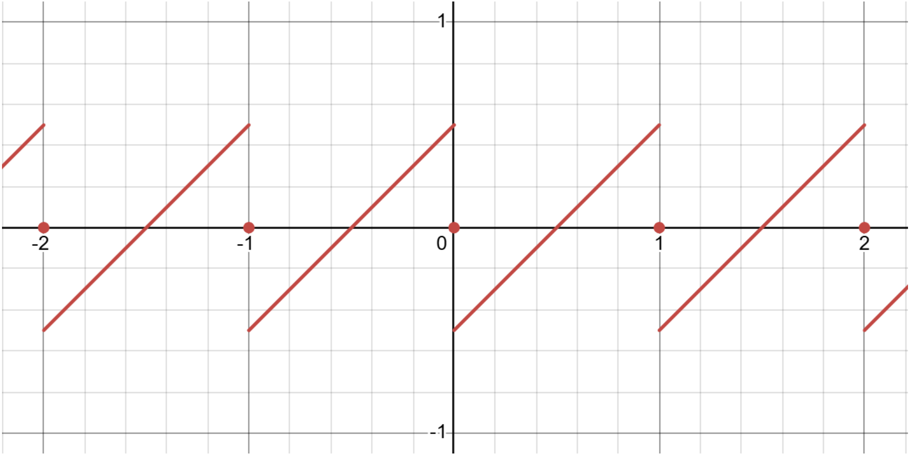
For coprime positive integers $h, k \in \mathbb{N}$, the classical Dedekind sum is defined by

$$s(h,k) = \sum_{j \pmod{k}} B_1\left(\frac{j}{k}\right) B_1\left(\frac{hj}{k}\right),$$

where $B_1(x)$ is the first Bernoulli function (also known as the sawtooth function):

$$B_1(x) = \begin{cases} 0, \ if \ x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2}, \ otherwise. \end{cases}$$

First Bernoulli Function



Congruence Subgroups

$SL_2(\mathbb{Z})$ is the group of invertible 2x2 matrices with integral values. Two important subgroups of $SL_2(\mathbb{Z})$ that we will need are

$$\Gamma_0(n) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod n \right\}$$

and

$$\Gamma_1(n) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : a, d \equiv 1 \bmod n, c \equiv 0 \bmod n \right\}.$$

These are called congruence subgroups and we usually denote their elements by γ. Note also that $\Gamma_1(n) \subseteq \Gamma_0(n)$.

Newform Dedekind Sum

It turns out that we can combine Dirichlet characters and the Dedekind eta function to arrive at a generalization of the Dedekind eta function. This generalization naturally leads to the generalized Dedekind sum.

Newform Dedekind Sum

It turns out that we can combine Dirichlet characters and the Dedekind eta function to arrive at a generalization of the Dedekind eta function. This generalization naturally leads to the generalized Dedekind sum.

Definition: Newform Dedekind Sum (SVY)

Let χ_1 and χ_2 be nontrivial primitive characters with moduli q_1 and q_2, respectively, such that $\chi_1\chi_2(-1) = 1$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$ with $c \geq 1$, the generalized Dedekind sum associated to χ_1 and χ_2 is given by

$$S_{\chi_1,\chi_2}(\gamma) = \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right).$$

Understanding the Two Conjecture

The Two Conjecture-*Motivations*

The main goal of this project is to *understand the image* of the generalized Dedekind sum. The following theorem is a helpful restriction on the image of the sum:

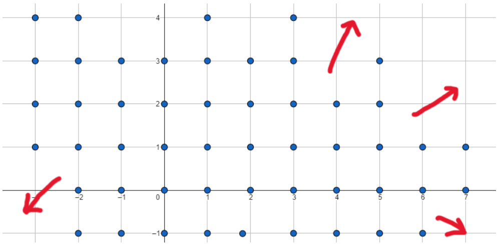
The Two Conjecture-*Motivations*

The main goal of this project is to *understand the image* of the generalized Dedekind sum. The following theorem is a helpful restriction on the image of the sum:

Theorem (Majure 2022)

The image of $S_{\chi_1,\chi_2}(\Gamma_1(q_1q_2))$ (note $\Gamma_1 \subseteq SL_2(\mathbb{Z})$) is a lattice of full rank inside F_{χ_1,χ_2}. In other words, $\text{Im}(S_{\chi_1,\chi_2}) = \{\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_n z_n : z_i \in \mathbb{Z},\ \alpha_i \in F\}$.

Example. When χ_1, χ_2 are quartic, $\text{Im}(S_{\chi_1,\chi_2})$ forms a lattice in $\mathbb{C}$.



Two Conjecture-intro

Conjecture (De Leon & McCormick)

Let χ_1, χ_2 be primitive, quadratic Dirichlet characters of moduli q_1 and q_2, respectively, such that $\chi_1\chi_2(-1) = 1$. Then,

$$S_{\chi_1,\chi_2}(\Gamma_1(q_1 q_2)) = 2\mathbb{Z}.$$

Two Conjecture-intro

Conjecture (De Leon & McCormick)

Let χ_1, χ_2 be primitive, quadratic Dirichlet characters of moduli q_1 and q_2, respectively, such that $\chi_1\chi_2(-1) = 1$. Then,

$$S_{\chi_1,\chi_2}(\Gamma_1(q_1q_2)) = 2\mathbb{Z}.$$

To prove this conjecture, we need to show

$$S_{\chi_1,\chi_2}(\Gamma_1(q_1q_2)) \subseteq 2\mathbb{Z}$$

and

$$2\mathbb{Z} \subseteq S_{\chi_1,\chi_2}(\Gamma_1(q_1q_2)).$$

The Two Conjecture with Other Characters

a.k.a. why 2 is my new favorite number

What if χ_1, χ_2 *are not quadratic?*
For χ of modulus q,

$$S_{\chi_1,\chi_2} = \sum\sum \overline{\chi_2}(\cdot)\overline{\chi_1}(\cdot) f \ g$$

where $\mathrm{Im}(f, g) \subseteq \mathbb{Q}$ and $\chi(\mathbf{n}) \subseteq \mathbb{Q}(\zeta_{\mathbf{k}})$ for $k = \varphi(q)$ and . This tells us that $\mathbf{S}_{\chi_1,\chi_2}(\mathbf{n}) \subseteq \mathbb{Q}(\zeta_{\mathbf{k}})$ as well. In other words, whatever **number field** χ_1, χ_2 live in, the *Dedekind sum* must live in the same number field!

The Two Conjecture with Other Characters

a.k.a. why 2 is my new favorite number

What if χ_1, χ_2 are not quadratic?
For χ of modulus q,

$$S_{\chi_1,\chi_2} = \sum \sum \overline{\chi_2}(\cdot)\overline{\chi_1}(\cdot) f \; g$$

where $\mathrm{Im}(f, g) \subseteq \mathbb{Q}$ and $\chi(\mathbf{n}) \subseteq \mathbb{Q}(\zeta_{\mathbf{k}})$ for $k = \varphi(q)$ and . This tells us that $\mathbf{S}_{\chi_1,\chi_2}(\mathbf{n}) \subseteq \mathbb{Q}(\zeta_{\mathbf{k}})$ as well. In other words, whatever **number field** χ_1, χ_2 live in, the *Dedekind sum* must live in the same number field!

In particular, the 2-conjecture for quadratic characters says that the $\mathrm{Im}(\mathbf{S}_{\chi_1,\chi_2})$ lies in *rational integers*. Now we wonder if the two conjecture still holds for more general characters. Let's compute some data!

thank you Carlos for the code!

The Two Conjecture with Other Characters

a.k.a. why 2 is my new favorite number

Computations for small values of q_1, q_2

$\chi_1 : mod(order)$	χ_2	Image	Codomain
$3(2)$	$7(2)$	$2\mathbb{Z}$	$\mathbb{Q}$
$5(2)$	$8(2)$	$2\mathbb{Z}$	$\mathbb{Q}$
$5(2)$	$7(3)$	$\{2a^* + 2\omega b : a, b \in \mathbb{Z}\}$	$\mathbb{Q}(\omega)$
$5(2)$	$7(3)$	$\{2a^* + 2\omega b : a, b \in \mathbb{Z}\}$	$\mathbb{Q}(\omega)$
$5(4)$	$5(4)$	$\{2a + 2ib : a, b \in \mathbb{Z}\}$	$\mathbb{Q}(i)$
$5(4)$	$5(4)$	$\{2a + 2ib : a, b \in \mathbb{Z}\}$	$\mathbb{Q}(i)$
$5(4)$	$5(4)$	$\{2a + 2ib : a, b \in \mathbb{Z}\}$	$\mathbb{Q}(i)$

*this has been verified for 1000 of the 7000 generating matrices

Evaluating the Dedekind Sum

What does it mean to look at the image of the Dedekind sum? First of all, we need a matrix $\gamma \in \Gamma_1$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the following conditions must be satisfied:

- a, b, c, d, q_1, $q_2 \in \mathbb{Z}$
- ad-bc $=$1
- gcd(a, c) $=$ 1
- $q_1 * q_2 | c$
- $a \equiv 1 \mod (q_1 * q_2)$

Evaluating the Dedekind Sum

Let $\gamma = \begin{pmatrix} 22 & x \\ 105 & x \end{pmatrix}$ with $q_1 = 3$ and $q_2 = 7$. Then the Dedekind sum is as follows:

$$S_{\chi_1,\chi_2}(\gamma) = \sum_{j \pmod{105}} \sum_{n \pmod{3}} \chi_2(j)\chi_1(n) B_1\left(\frac{j}{105}\right) B_1\left(\frac{n}{3} + \frac{22j}{105}\right).$$

From our computations, this sum surprisingly evaluates to 2.

The floor function in the Bernoulli function involved in this sum and the Dirichlet characters make it inherently difficult to evaluate.

Bounding the Denominator

New way to write the sum

One way to re-write the Dedekind sum is

$$\sum_{n \bmod q_1} \sum_{j_0 \bmod q_2} \overline{\chi_1}(n)\overline{\chi_2}(j_0)\left(\sum_{k \bmod (c/q_2)} B_1\left(\frac{j_0}{c} + \frac{q_2 k}{c}\right) B_1\left(\frac{n}{q_1} + \frac{a j_0}{c} + \frac{a k q_2}{c}\right)\right)$$

Because of our previous observations, it seems like the Bernoulli functions may be controlling the integrality of the sum, and the characters control which field the sum lies in. Thus, as we continue, we will work on showing the Bernoulli functions on the inside are integers so we can show the whole sum becomes an integer.

Properties of the Bernoulli Function

In order to understand the sum better, we needed to understand properties of the Bernoulli function.

Proposition (Rademacher 1956)

Let $y \in \mathbb{R}$, $\lambda, k \in \mathbb{Z}$, and B_1 be the Bernoulli Sawtooth function.

- $\sum_{\lambda \bmod k} B_1\left(\frac{x+\lambda}{k}\right) = B_1(x)$. In particular, if $x \in \mathbb{Z}$, the sum is zero.
- $\sum_{\lambda \bmod k} \lambda B_1\left(\frac{\lambda}{k}\right) = \frac{1}{6}(k-1)(2k-1)$.

Using these properties, we derived the following properties.

Properties of the Bernoulli Function

In order to understand the sum better, we needed to understand properties of the Bernoulli function.

Proposition (Rademacher 1956)

Let $y \in \mathbb{R}$, $\lambda, k \in \mathbb{Z}$, and B_1 be the Bernoulli Sawtooth function.

- $\sum_{\lambda \text{ mod } k} B_1\left(\frac{x+\lambda}{k}\right) = B_1(x)$. In particular, if $x \in \mathbb{Z}$, the sum is zero.
- $\sum_{\lambda \text{ mod } k} \lambda B_1\left(\frac{\lambda}{k}\right) = \frac{1}{6}(k-1)(2k-1)$.

Using these properties, we derived the following properties.

Proposition: Bernoulli Properties

- Let $x \in \mathbb{R}$ and $a \in \mathbb{Z}$ such that $gcd(a,k) = 1$. Then

$$\sum_{\lambda \text{ mod } k} B_1\left(\frac{a\lambda + x}{k}\right) = \sum_{\lambda \text{ mod } k} B_1\left(\frac{\lambda + x}{k}\right)$$

- Let $y \in \mathbb{R}, [y] \equiv m$ mod k. Then

$$\sum_{\lambda \text{ mod } k} \lambda B_1\left(\frac{y + \lambda}{k}\right) = f_k(\{y\}) - \frac{mk}{2} + \frac{m^2}{2}$$

with $f_k(\{y\}) = (k-1)(6\{y\} + k - 2)/12$.

Formula for the Newform Dedekind Sum

Proposition: Newform Dedekind Sum Formula

If $c = rq_1q_2$, then we can write the newform Dedekind Sum as

$$S(a,c) = -\frac{1}{rq_1}\sum_{j=0}^{c-1}\sum_{n=0}^{q_1-1}\overline{\chi_2}(j_0)\overline{\chi_1}(n)\left\lfloor\frac{j}{q_2}\right\rfloor\left\lfloor\frac{aj}{c}+\frac{n}{q_1}\right\rfloor$$

Ignoring the factor in front, the double sum is an integer, this means the denominator of $S(a,c)$ divides rq_1.

Formula for the Newform Dedekind Sum

We begin by rewriting the newform Dedekind Sum in terms of the fractional part function, which we denote using curly brackets

$$S(a,c) = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \left\{ \frac{j}{c} \right\} \left\{ \frac{aj}{c} + \frac{n}{q_1} \right\}$$

Formula for the Newform Dedekind Sum

We begin by rewriting the newform Dedekind Sum in terms of the fractional part function, which we denote using curly brackets

$$S(a,c) = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \left\{\frac{j}{c}\right\} \left\{\frac{aj}{c} + \frac{n}{q_1}\right\}$$

Substituting $\left\{\frac{j}{c}\right\} = \frac{j}{c}$ and $\left\{\frac{aj}{c} + \frac{n}{q_1}\right\} = \left(\frac{aj}{c} + \frac{n}{q_1}\right) - \left\lfloor \frac{aj}{c} + \frac{n}{q_1} \right\rfloor$ yields $S(a,c) = P - Q$ where

$$P = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \frac{j}{c} \left(\frac{aj}{c} + \frac{n}{q_1}\right)$$

$$Q = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \frac{j}{c} \left\lfloor \frac{aj}{c} + \frac{n}{q_1} \right\rfloor$$

Formula for the Newform Dedekind Sum

Let's look at the first double sum

$$P = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \left(\frac{aj^2}{c^2} + \frac{nj}{cq_1} \right)$$

Formula for the Newform Dedekind Sum

Let's look at the first double sum

$$P = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \left(\frac{aj^2}{c^2} + \frac{nj}{cq_1} \right)$$

Using the orthogonality of Dirichlet characters, we can see that the $\frac{aj^2}{c^2}$ term will simply vanish.

$$P = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \frac{nj}{cq_1}$$

Formula for the Newform Dedekind Sum

Let's look at the first double sum

$$P = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \left(\frac{aj^2}{c^2} + \frac{nj}{cq_1} \right)$$

Using the orthogonality of Dirichlet characters, we can see that the $\frac{aj^2}{c^2}$ term will simply vanish.

$$P = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \frac{nj}{cq_1}$$

Using the periodicity of χ_2 and a change of variables, we get

$$P = \frac{1}{q_1 q_2} \left[\sum_{m=0}^{q_2-1} \overline{\chi_2}(m) m \right] \cdot \left[\sum_{n=0}^{q_1-1} \overline{\chi_1}(n) n \right]$$

Formula for the Newform Dedekind Sum

Recalling that $S(a,c) = P - Q$, let's look at Q. We will express $Q = R + T$, with $R = P$, so it will turn out that $S(a,c) = -T$.

$$Q = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n)\frac{j}{c}\left\lfloor \frac{aj}{c} + \frac{n}{q_1} \right\rfloor$$

Formula for the Newform Dedekind Sum

Recalling that $S(a,c) = P - Q$, let's look at Q. We will express $Q = R + T$, with $R = P$, so it will turn out that $S(a,c) = -T$.

$$Q = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n)\frac{j}{c}\left\lfloor \frac{aj}{c} + \frac{n}{q_1} \right\rfloor$$

We can use a change of variables $j \to j_0 + kq_2$ to obtain

$$Q = \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} \frac{j_0 + kq_2}{c} \left\lfloor \frac{a(j_0 + kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

Formula for the Newform Dedekind Sum

Recalling that $S(a,c) = P - Q$, let's look at Q. We will express $Q = R + T$, with $R = P$, so it will turn out that $S(a,c) = -T$.

$$Q = \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \frac{j}{c} \left\lfloor \frac{aj}{c} + \frac{n}{q_1} \right\rfloor$$

We can use a change of variables $j \to j_0 + kq_2$ to obtain

$$Q = \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} \frac{j_0 + kq_2}{c} \left\lfloor \frac{a(j_0 + kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

We can distribute this sum to get

$$R = \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} \frac{j_0}{c} \left\lfloor \frac{a(j_0 + kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

$$T = \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} \frac{kq_2}{c} \left\lfloor \frac{a(j_0 + kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

Formula for the Newform Dedekind Sum

$$R = \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} \frac{j_0}{c} \left\lfloor \frac{a(j_0+kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

Formula for the Newform Dedekind Sum

$$R = \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} \frac{j_0}{c} \left\lfloor \frac{a(j_0+kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

After rewriting some terms and simplifying this sum using the formula

$$\sum_{\lambda=0}^{k-1} \left\lfloor \frac{x+a\lambda}{k} \right\rfloor = \lfloor x \rfloor + \frac{1}{2}(a-1)(k-1)$$

Formula for the Newform Dedekind Sum

$$R = \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} \frac{j_0}{c} \left\lfloor \frac{a(j_0+kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

After rewriting some terms and simplifying this sum using the formula

$$\sum_{\lambda=0}^{k-1} \left\lfloor \frac{x+a\lambda}{k} \right\rfloor = \lfloor x \rfloor + \frac{1}{2}(a-1)(k-1)$$

we get

$$R = \frac{1}{q_1q_2} \left[\sum_{m=0}^{q_2-1} \overline{\chi_2}(m)m\right] \cdot \left[\sum_{n=0}^{q_1-1} \overline{\chi_1}(n)n\right] = P$$

Formula for the Newform Dedekind Sum

$$R = \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} \frac{j_0}{c} \left\lfloor \frac{a(j_0+kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

After rewriting some terms and simplifying this sum using the formula

$$\sum_{\lambda=0}^{k-1} \left\lfloor \frac{x+a\lambda}{k} \right\rfloor = \lfloor x \rfloor + \frac{1}{2}(a-1)(k-1)$$

we get

$$R = \frac{1}{q_1 q_2} \left[\sum_{m=0}^{q_2-1} \overline{\chi_2}(m)m \right] \cdot \left[\sum_{n=0}^{q_1-1} \overline{\chi_1}(n)n \right] = P$$

Since these two sums will cancel out, we get

$$S(a,c) = -T$$

Formula for the Newform Dedekind Sum

Writing out $S(a,c) = -T$, we have

$$S(a,c) = -\frac{1}{rq_1} \sum_{j_0=0}^{q_2-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j_0)\overline{\chi_1}(n) \sum_{k=0}^{rq_1-1} k \left\lfloor \frac{a(j_0+kq_2)}{c} + \frac{n}{q_1} \right\rfloor$$

Finally, we revert the change of variables, using $k = \left\lfloor \frac{j}{q_2} \right\rfloor$ to get

$$S(a,c) = -\frac{1}{rq_1} \sum_{j=0}^{c-1} \sum_{n=0}^{q_1-1} \overline{\chi_2}(j)\overline{\chi_1}(n) \left\lfloor \frac{j}{q_2} \right\rfloor \left\lfloor \frac{aj}{c} + \frac{n}{q_1} \right\rfloor$$

If we ignore the factor outside, the double sum is an integer. So, we obtain that the denominator of $S(a,c)$ divides rq_1.

Reciprocity Law & Homomorphism

The newform Dedekind sum satisfies a family of reciprocity laws, we have found the following one the most useful.

Proposition: Reciprocity Law (SVY20)

Let $\gamma = \begin{pmatrix} a & b \\ cq_1q_2 & d \end{pmatrix} \in \Gamma_1(q_1q_2)$ and $\gamma' = \begin{pmatrix} d & -c \\ -bq_1q_2 & a \end{pmatrix}$.

Then,

$$S_{\chi_1,\chi_2}(\gamma) = \chi_1(-1)S_{\chi_2,\chi_1}(\gamma').$$

Proposition: Homomorphism Property (SVY20)

$S_{\chi_1,\chi_2} : \Gamma_1(q_1q_2) \to \mathbb{C}$ is a group homomorphism.

Newform Dedekind Sum Denominator

Using the fact that S_{χ_1,χ_2} is a group homomorphism, we can see that

$$S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\right) = S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & an+b \\ c & cn+d \end{bmatrix}\right)$$

Newform Dedekind Sum Denominator

Using the fact that S_{χ_1,χ_2} is a group homomorphism, we can see that

$$S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\right) = S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & an+b \\ c & cn+d \end{bmatrix}\right)$$

Then, by the Reciprocity Law,

$$S_{\chi_1,\chi_2}(a,c) = \pm S_{\chi_2\chi_1}(cn+d, -(an+b)q_1q_2)$$

Newform Dedekind Sum Denominator

Using the fact that S_{χ_1,χ_2} is a group homomorphism, we can see that

$$S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\right) = S_{\chi_1,\chi_2}\left(\begin{bmatrix} a & an+b \\ c & cn+d \end{bmatrix}\right)$$

Then, by the Reciprocity Law,

$$S_{\chi_1,\chi_2}(a,c) = \pm S_{\chi_2\chi_1}(cn+d, -(an+b)q_1q_2)$$

Since a and b are coprime, by Dirichlet's Theorem on Arithmetic Progressions, there are infinitely many primes of the form $an+b$.

Newform Dedekind Sum Denominator

Letting d be the denominator of $S_{\chi_1,\chi_2}(a,c)$, we can choose two primes p_1 and p_2 such that

$$d \mid p_1 q_2 \text{ and } d \mid p_2 q_2, \text{ so } d \mid q_2$$

Newform Dedekind Sum Denominator

Letting d be the denominator of $S_{\chi_1,\chi_2}(a,c)$, we can choose two primes p_1 and p_2 such that

$$d \mid p_1 q_2 \text{ and } d \mid p_2 q_2 \text{, so } d \mid q_2$$

Using the reciprocity formula and repeating the argument, we get that

$$d \mid q_1$$

Newform Dedekind Sum Denominator

Letting d be the denominator of $S_{\chi_1,\chi_2}(a,c)$, we can choose two primes p_1 and p_2 such that

$$d \mid p_1 q_2 \text{ and } d \mid p_2 q_2 \text{, so } d \mid q_2$$

Using the reciprocity formula and repeating the argument, we get that

$$d \mid q_1$$

So,

$$d \mid q_1, q_2 \text{ and thus } d \mid gcd(q_1, q_2)$$

Newform Dedekind Sum Denominator

Letting d be the denominator of $S_{\chi_1,\chi_2}(a,c)$, we can choose two primes p_1 and p_2 such that

$$d \mid p_1 q_2 \text{ and } d \mid p_2 q_2, \text{ so } d \mid q_2$$

Using the reciprocity formula and repeating the argument, we get that

$$d \mid q_1$$

So,

$$d \mid q_1, q_2 \text{ and thus } d \mid gcd(q_1, q_2)$$

Proposition: Newform Dedekind Sum Denominator

The denominator of $S_{\chi_1,\chi_2}(a,c)$ divides $gcd(q_1,q_2)$. So, when q_1 and q_2 are coprime, $S_{\chi_1,\chi_2}(a,c)$ is an integer.

Advancements Toward Evenness

Symmetry in the Generalized Dedekind Sum

Proposition

Let χ_1 and χ_2 be nontrivial, primitive, quadratic characters such that $\chi_1\chi_2(-1) = 1$. The terms of the generalized Dedekind sum over $j \pmod{c}$ demonstrate symmetry across the term j $= \frac{c}{2}$ such that

$$\overline{\chi_2}(j)\overline{\chi_1}(n)B_1\left(\frac{j}{c}\right)B_1\left(\frac{n}{q_1}+\frac{aj}{c}\right) = \overline{\chi_2}(c-j)\overline{\chi_1}(n)B_1\left(\frac{c-j}{c}\right)B_1\left(\frac{n}{q_1}+\frac{a(c-j)}{c}\right)$$

Once we have a way to show the sum is an integer, this could help prove that the sum is even!

Symmetry in the sum

Beginning with a change of variables $j \to c - j$ and $n \to -n$, we have

$$\overline{\chi_2}(c-j)\overline{\chi_1}(-n)B_1\Big(\frac{c-j}{c}\Big)B_1\Big(\frac{a(c-j)}{c}+\frac{-n}{q_1}\Big). \tag{1}$$

Since the characters are quadratic, they are not complex. Thus $\overline{\chi} = \chi$. Also since $\chi_1\chi_2(-1) = 1$, the characters have the same parity. In the case where the parity of the characters is odd, by periodicity,

$$\chi_2(c-j) = -\chi_2(j),$$

$$\chi_1(-n) = -\chi_1(n),$$

so

$$\chi_2(c-j)\chi_1(-n) = \chi_2(j)\chi_1(n).$$

Symmetry in the sum

In the case where the parity of the characters in even, by periodicity,

$$\chi_2(c-j) = \chi_2(j),$$

$$\chi_1(-n) = \chi_1(n),$$

so

$$\chi_2(c-j)\chi_1(-n) = \chi_2(j)\chi_1(n).$$

Thus we have

$$\chi_2(j)\chi_1(n)B_1\Big(1-\frac{j}{c}\Big)B_1\Big(a-(\frac{aj}{c}+\frac{n}{q_1})\Big),$$

which can be further simplified using the periodic and odd properties of the Bernoulli Sawtooth function.

Condition for an even sum

We combine our symmetry and integral observations to show that the sum is an even integer. First, we need the following Lemma.

Lemma

Let $S_{\chi_1,\chi_2} = (a, c)$*. For* $c = rq_1q_2$*, we can assume* r *is odd.*

In order to prove this, we use the reciprocity formula and homomorphism properties of the Dedekind Sum. The following theorem is a consequence of this Lemma.

Condition for an even sum

We combine our symmetry and integral observations to show that the sum is an even integer. First, we need the following Lemma.

Lemma

Let $S_{\chi_1,\chi_2} = (a, c)$. *For* $c = rq_1q_2$, *we can assume* r *is odd.*

In order to prove this, we use the reciprocity formula and homomorphism properties of the Dedekind Sum. The following theorem is a consequence of this Lemma.

Theorem

If both q_1 *and* q_2 *are odd, then*

$$S_{\chi_1,\chi_2}(a, c) \subseteq \frac{1}{rq_1} 2\mathbb{Z}[\chi_1, \chi_2]$$

Condition for an even sum

Theorem

If both q_1 and q_2 are odd, then

$$S_{\chi_1,\chi_2}(a,c) \subseteq \frac{1}{rq_1} 2\mathbb{Z}[\chi_1,\chi_2]$$

$$\text{By symmetry} \implies S_{\chi_1,\chi_2}(a,c) = \frac{2A}{c^2}. A \in \mathbb{Z}$$

$$\text{By sum manipulation} \implies S_{\chi_1,\chi_2}(a,c) = \frac{B}{rq_1}, B \in \mathbb{Z}$$

Equating these, and assuming c is odd, we get

$$2Arq_1 = c^2 B,$$

so B is even.

Next steps

- Use the reciprocity formula to rewrite the innermost sum
- Understand how the a value affects Bernoulli and floor functions
- We know $a = 1$ is in the kernel, look for other kernel values we can find
- Pursue the possibility of a new reciprocity formula

References

- [DM23]E. De Leon and W. McCormick. “Image of newform Dedekind sums attached to quadratic characters”. unpublished. 2023
- [Maj23]Mitchell Majure. Algebraic properties of the values of newform Dedekind sums. 2023
- [RG72]Hans Rademacher and Emil Grosswald. Dedekind sums. No. 16. Mathematical Association of America, Washington, D.C., 1972
- [SVY20]T. Stucker, A. Vennos, and M. P. Young. “Dedekind sums arising from newform Eisenstein series”. In: International Journal of Number Theory 16.10 (Aug. 2020), pp. 2129–2139. ISSN: 1793-7310.

Thank You!

We would like to thank Dr. Matthew Young, Dr. Agniva Dasgupta, the organizers of the REU, Texas A&M, and NSF.