# Comparative Analyses of the Type D ASEP: Stochastic Fusion and Crystal Bases

Eva Engel, Connor Panish, Lillian Stolberg, Erik Brodsky

Texas A&M REU 2024
advised by Professor Jeffrey Kuan

July 25, 2024

# Markov Processes

A **Markov process** is a collection of random variables taken over continuous time such that the *future* state of the process depends only on the *present* state of the process [Kua].

The **state space** of a Markov process is the collection of all values that the process can take.

The **transition matrix** of a Markov process lists the probabilities of jumps between different states.

$$\begin{array}{c} \begin{matrix} \quad a & \quad\; b & \quad\; c \end{matrix} \\ \begin{pmatrix} 0.1 & 0.7 & 0.2 \\ 0.25 & 0.25 & 0.5 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} a \\ b \\ c \end{matrix} \end{array}$$

# Generators

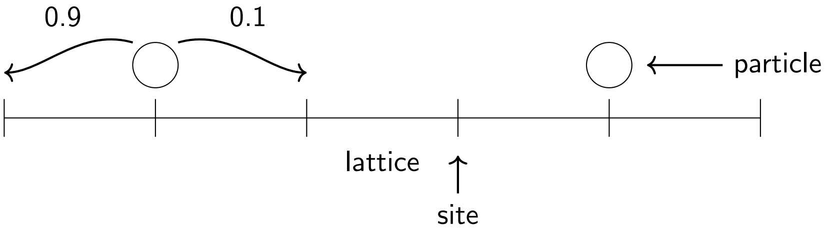
Take Markov process $X(t)$ with transition matrix $P(t)$. Note that $P(0) = I$.

The **generator** matrix $Q$ describes $X(t)$ and can be found by taking $P(t) = e^{tQ}$ or $P'(O) = QP(0) = Q$ [Fer].

# ASEP

An **asymmetric simple exclusion process** (ie. ASEP) is a Markov process in which particles on a lattice jump between sites [Spi70].

One particle is allowed at each site, and the jump rates of particles to the left and right are different such that the particle system exhibits net drift in one direction.



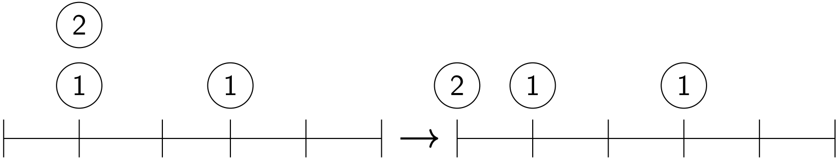
# Type D ASEP

**Type D ASEP** extends ASEP to two particle species (ie. types of particles), such that two particles of different species can exist at the same site but two particles of the same species cannot [RLY23].

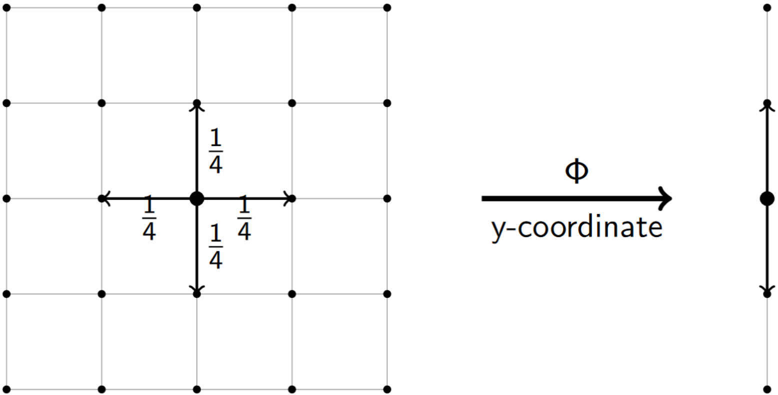
This process takes three conditions: an asymmetry parameter that describes the direction of particle drift, a parameter that gives the speed of particle drift, and a parameter that describes how particles of different species interact.

# Visualizing Type D ASEP

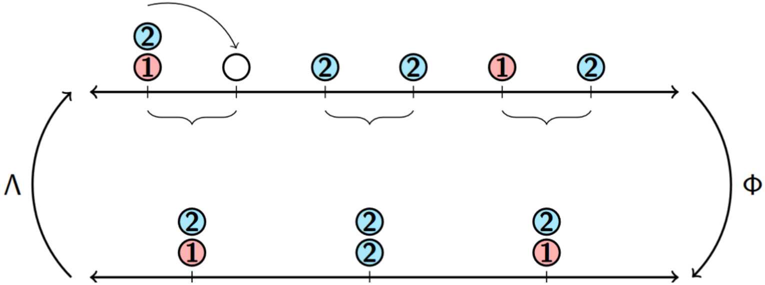
Jump rates vary by particle species (ie. species 1 or 2) and orientation.



# Markov Projection



# Stochastic Fusion




[Kua19]

# The Problem


$$\mathbf{Q_t} = \mathbf{\Lambda P_t \Phi}$$

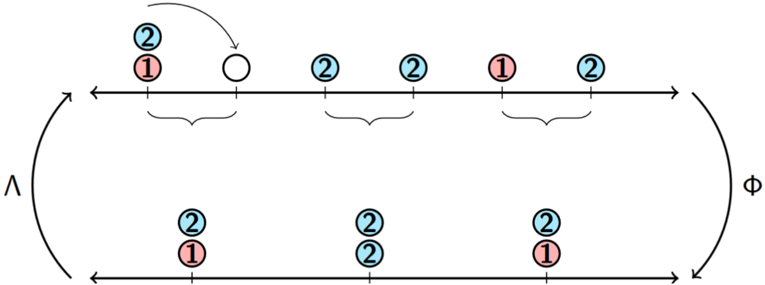
$Q_t$ : Fused Transition Matrix

$\Lambda$ : Reverse Fusion Map

$P_t$ : Original Transition Matrix

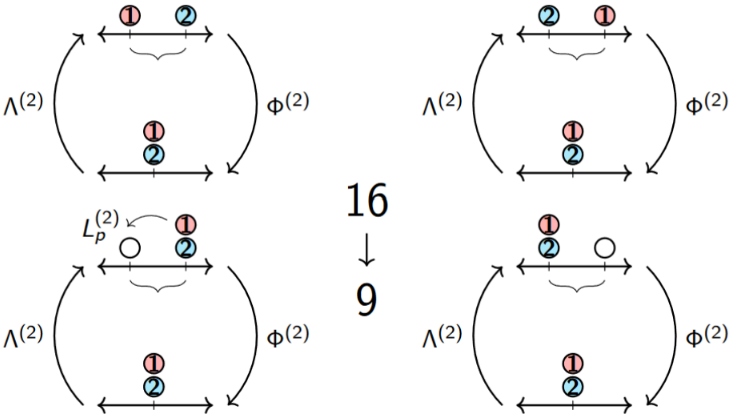
$\Phi$ : Fusion Map

# The Problem Continued

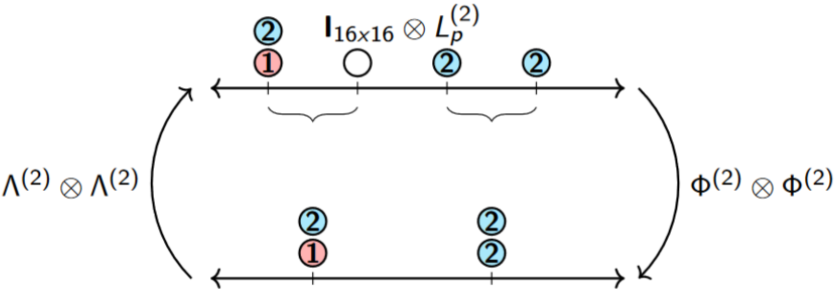



$$Q_t = \Lambda P_t \Phi$$
$$L_Q = \Lambda L_p \Phi$$

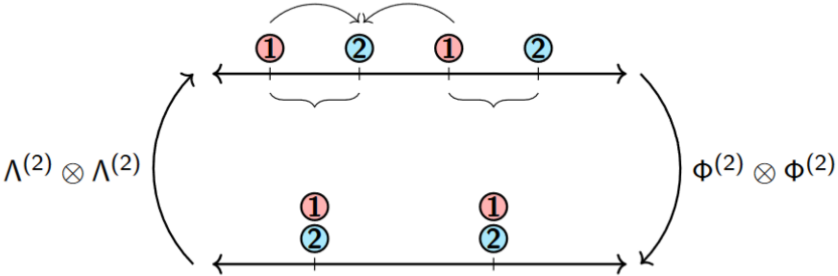
# The Most Basic Case



# Expanding to Four Lattice Sites



# Middle Swap

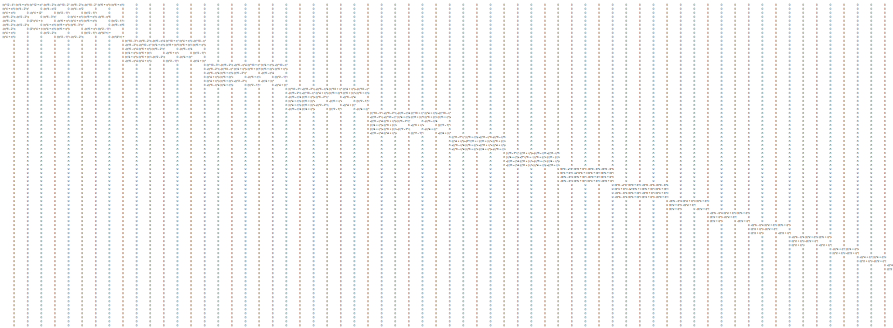


# Making Fused Matrix

$$L_Q = \Lambda L_p \Phi$$

$$L_Q = (\Lambda^{(2)} \otimes \Lambda^{(2)})(\mathbf{I}_{16\times 16} \otimes L_2)(\Phi^{(2)} \otimes \Phi^{(2)})$$

# Displaying Fused Matrix



$$L_Q = \mathcal{L}_9 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_6 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_4 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_3 \oplus \bigoplus_{i=1}^{8} \mathcal{L}_2 \oplus \bigoplus_{i=1}^{4} [0]$$

# Three Block of Fused Matrix

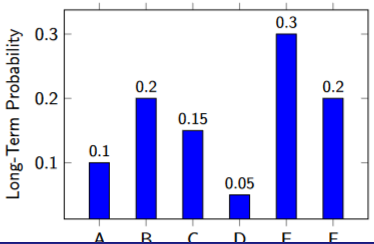
$$\begin{bmatrix} * & \frac{q^2+q^{4n}}{(q^6+2q^4+q^2)q^{2n}} & \frac{q^6+q^{4n+4}}{(q^4+2q^2+1)q^{2n}} \\ \frac{q^2+q^{4n}}{q^{2n}} & * & 0 \\ \frac{q^2+q^{4n}}{q^{2n+2}} & 0 & * \end{bmatrix}$$

$$L_Q^D = \mathcal{L}_9 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_6 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_4 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_3 \oplus \bigoplus_{i=1}^{8} \mathcal{L}_2 \oplus \bigoplus_{i=1}^{4} [0]$$

# Defining the Stationary Distribution

The **stationary distribution** of a Markov chain describes the distribution of states visited by the process in the long-term [Mar06].

Rigorously, let $X_t$ be a Markov chain with transition matrix $P$. The stationary distribution $\pi$ of $X_t$ satisfies $\pi P = \pi$ [Mar06]. For generator $L$, $\pi L = 0$.



# Finding Stationary Distributions

A **communicating class** is a set of states that can transform into each other via Type D ASEP.

We find the stationary distribution for each communicating class. For communicating class $A$ with stochastic fusion generator $L_{Q_A}$, the stationary distribution $\pi_A$ for $A$ satisfies $\pi_A L_{Q_A} = 0$.

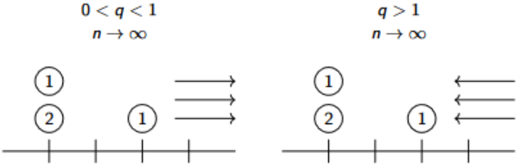
Stationary distribution is *not* affected by the speed of convergence.

# Taking $n \to \infty$

Recall: $q$ measures the direction of drift of the system and $n$ measures the speed of drift.

Multiply generator of stochastic fusion process by $q^{\pm 2n}$ then take limit as $n \to \infty$. We get a *finite* limit.

Intuitively, as drift speed of Type D ASEP increases to infinity, the drift speed of stochastically-fused process also increases to infinity.



# Spectral Gap

$\lambda_2$ = second-largest eigenvalue of generator matrix

**Spectral gap** is $|\lambda_2|$ for our project.

For $q > 1$, $\lim_{n\to\infty} |\lambda_2| = \infty$ for communicating classes with 2 or 3 states.

For $0 < q < 1$, $\lim_{n\to\infty} |\lambda_2| = 0$ for communicating classes with 2 or 3 states.

# Relaxation Time

**Relaxation time** is $\frac{1}{|\lambda_2|}$ [Bli64].

For $q > 1$, $\lim_{n\to\infty} \frac{1}{|\lambda_2|} = 0$, so speed of convergence to stationary distribution accelerates for 2 and 3-state communicating classes.

For $0 < q < 1$, $\lim_{n\to\infty} \frac{1}{|\lambda_2|} = \infty$, so speed of convergence to stationary distribution becomes very slow for 2 and 3-state communicating classes.

# Defining Markov Duality

Take Markov processes $X_t$ and $Y_t$ with state spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $L_X$ and $L_Y$ be generators of $X_t$ and $Y_t$. Let $M$ be a matrix representing a function.

$X_t$ and $Y_t$ are **dual** with respect to the function represented by $M$ if $L_X M = M L_Y^T$ [DF90].

$$\underbrace{\begin{bmatrix} L_X(x_1,x_1) & L_X(x_1,x_2) & L_X(x_1,x_3) \\ L_X(x_2,x_1) & L_X(x_2,x_2) & L_X(x_2,x_3) \\ L_X(x_3,x_1) & L_X(x_3,x_2) & L_X(x_3,x_3) \end{bmatrix}}_{\mathbf{L_X}} \underbrace{\begin{bmatrix} M(x_1,y_1) & M(x_1,y_2) & M(x_1,y_3) \\ M(x_2,y_1) & M(x_2,y_2) & M(x_2,y_3) \\ M(x_3,y_1) & M(x_3,y_2) & M(x_3,y_3) \end{bmatrix}}_{\mathbf{M}}$$

$$=$$

$$\underbrace{\begin{bmatrix} M(x_1,y_1) & M(x_1,y_2) & M(x_1,y_3) \\ M(x_2,y_1) & M(x_2,y_2) & M(x_2,y_3) \\ M(x_3,y_1) & M(x_3,y_2) & M(x_3,y_3) \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} L_Y(y_1,y_1) & L_Y(y_2,y_1) & L_Y(y_3,y_1) \\ L_Y(y_1,y_2) & L_Y(y_2,y_2) & L_Y(y_3,y_2) \\ L_Y(y_1,y_3) & L_Y(y_2,y_3) & L_Y(y_3,y_3) \end{bmatrix}}_{\mathbf{L_Y^T}}$$

# q-Krawtchouk Polynomials

We are looking for **self-duality**, ie. $D$ such that $L_Q^D D = D(L_Q^D)^T$.

[Bly+23] proposed a self-duality function for *unfused* Type D ASEP using q-Krawtchouk polynomials.

Does this function work for our *fused* Type D ASEP?

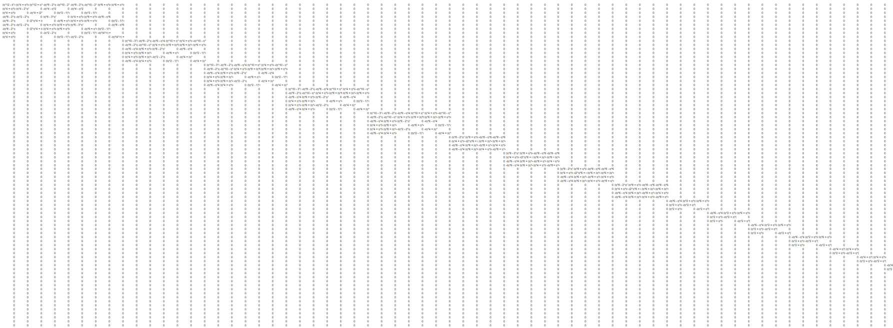
# q-Krawtchouk Continued

For the $D$ we found using [Bly+23]'s method, $L_Q^D D[1,2] \neq D(L_Q^D)^T[1,2]$.

So $D$ is **not** a matrix of Markov self-duality.

Using a probabilistic approach rather than an algebraic one created different duality functions.

# Recall Block Diagonal



$$L_Q^D = \mathcal{L}_9 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_6 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_4 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_3 \oplus \bigoplus_{i=1}^{8} \mathcal{L}_2 \oplus \bigoplus_{i=1}^{4} [0]$$

# Trivial Duality and Duality Reveiwed

$$E_x(\mathcal{D}(x, X(t)) = E_x(\mathcal{D}(X(t), x))$$

$$L_Q^D \mathcal{D} = \mathcal{D}(L_Q^D)^T$$

# Diagonalization

$$\mathcal{Z} = \bigoplus_{i=1}^{4} [0]$$

$$L_Q^D = \mathcal{L}_9 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_6 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_4 \oplus \bigoplus_{i=1}^{4} \mathcal{L}_3 \oplus \bigoplus_{i=1}^{8} \mathcal{L}_2 \oplus \mathcal{Z}$$

$$= \mathcal{P}_9 A_9 \mathcal{P}_9^{-1} \oplus \bigoplus_{i=1}^{4} \mathcal{P}_6 A_6 \mathcal{P}_6^{-1} \oplus \bigoplus_{i=1}^{4} \mathcal{P}_4 A_4 \mathcal{P}_4^{-1}$$

$$\oplus \bigoplus_{i=1}^{4} \mathcal{P}_3 A_3 \mathcal{P}_3^{-1} \oplus \bigoplus_{i=1}^{8} \mathcal{P}_2 A_2 \mathcal{P}_2^{-1} \oplus \mathcal{Z}$$

$$= \mathcal{P} \mathcal{A} \mathcal{P}^{-1} \oplus \mathcal{Z}$$

# Finding a Dual

$$L_Q^D \mathcal{D} = \mathcal{D}(L_Q^D)^T$$

$$L_Q^D = \mathcal{P}\mathcal{A}\mathcal{P}^{-1} \oplus \mathcal{Z}$$

$$\mathcal{D} = \mathcal{P}(\mathcal{P})^T \oplus \mathcal{Z}$$

$$= \mathcal{P}_9(\mathcal{P}_9)^T \oplus \bigoplus_{i=1}^{4} \mathcal{P}_6(\mathcal{P}_6)^T \oplus \bigoplus_{i=1}^{4} \mathcal{P}_4(\mathcal{P}_4)^T$$

$$\oplus \bigoplus_{i=1}^{4} \mathcal{P}_3(\mathcal{P}_3)^T \oplus \bigoplus_{i=1}^{8} \mathcal{P}_2(\mathcal{P}_2)^T \oplus \mathcal{Z}$$

# Non-Trivial Dual Example

$$\mathcal{P}_4(\mathcal{P}_4)^T =$$

$$\begin{bmatrix} 1+\frac{1}{q^8}+\frac{1}{q^{16}} & 1-\frac{q^4-1}{q^8}-\frac{1}{q^{12}} & 1-\frac{1}{q^{12}} & 1-\frac{1}{q^4}+\frac{1}{q^8} \\ 1-\frac{q^4-1}{q^8}-\frac{1}{q^{12}} & 2+\frac{\left(q^4-1\right)^2}{q^8}+\frac{1}{q^8} & \frac{1}{q^8} & 1+\frac{q^4-1}{q^4}-\frac{1}{q^4} \\ 1-\frac{1}{q^{12}} & \frac{1}{q^8} & 2+\frac{1}{q^8} & 1-\frac{1}{q^4} \\ 1-\frac{1}{q^4}+\frac{1}{q^8} & 1+\frac{q^4-1}{q^4}-\frac{1}{q^4} & 1-\frac{1}{q^4} & 3 \end{bmatrix}$$

$$L_Q^D \mathcal{D} = \mathcal{D}(L_Q^D)^T$$

# Lie Groups

A **Lie group** is a set $G$ which is structurally both a group and a differentiable manifold.

- Multiplication, $m : G \times G \to G$, is a differentiable map.
- Inversion is a smooth map.

# Examples of Lie Groups

- $(\mathbb{R}^n, +)$
- $(\mathbb{R}^*, \times)$
- $(S^1, \times)$
- $(GL_n(\mathbb{R}), \times)$ where $GL_n(\mathbb{R})$ is embedded in $\mathbb{R}^{n^2}$
- Many matrix groups when embedded

# Lie Algebras

A **Lie algebra** is the tangent space to a Lie group at the identity of the group.

# Lie Algebras in $GL_n(\mathbb{R})$

For a Lie group $G \subset GL_n(\mathbb{R})$, the corresponding Lie algebra $\mathfrak{g}$ is a vector space such that, for some neighborhood $U \subset GL_n(\mathbb{R})$ of $1_G$ and some neighborhood $\mathfrak{u} \subset \mathfrak{gl}_n(\mathbb{R})$ of $0_{\mathfrak{g}}$, the following maps are inverses of each other:

- $\log : U \cap G \to \mathfrak{u} \cap \mathfrak{g}$
- $\exp : \mathfrak{u} \cap \mathfrak{g} \to U \cap G$

# Examples of Lie Algebras

| Lie group | Lie algebra |
|---|---|
| $GL_n(\mathbb{R})$ | $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ |
| $SL_n(\mathbb{R})$ | $\mathfrak{sl}_n(\mathbb{R}) = \{x \in GL_n(\mathbb{R}) \vert \mathrm{tr}(x) = 0\}$ |
| $SO_n(\mathbb{R})$ | $\mathfrak{so}_n(\mathbb{R}) = \{x \in GL_n(\mathbb{R}) \vert x + x^T = 0\}$ |

# The Special Orthogonal Group, $SO_{2n}$

$$SO_{2n} = \left\{X \in M_{2n\times 2n}(\mathbb{C}) : XX^T = I, \det X = 1\right\}$$

# The Special Orthogonal Lie Algebra

$$\mathfrak{so}_{2n} = \left\{ \begin{bmatrix} A & C \\ -C^T & B \end{bmatrix} : A, B, C \in M_{n\times n}(\mathbb{C}), A = -A^T, B = -B^T \right\}$$

# Universal Enveloping Algebra

In a Lie algebra $\mathfrak{g}$, the usual multiplication is not well-defined in general.
Therefore, we define the **universal enveloping algebra**, $\mathcal{U}(\mathfrak{g})$, to be an algebra generated by elements in $\mathfrak{g}$ which follows certain relations, including a commutator relation.

# Universal Enveloping Algebra — $\mathcal{U}(\mathfrak{so}_6)$

$\mathcal{U}(\mathfrak{so}_6)$ is generated by $\{E_1, E_2, E_3, F_1, F_2, F_3, H_1, H_2, H_3\}$ with relations:

$$[E_i, F_i] = H_i, \qquad 1 \leq i \leq 3$$

and

$$E_l^2 E_j + E_j E_l^2 = 2E_l E_j E_l; \quad F_l^2 F_j + F_j F_l^2 = 2F_l F_j F_l$$

for $(l, j) \in \{(1,2), (2,1), (1,3), (3,1)\}$ .

# The Quantum Group — $\mathcal{U}_q(\mathfrak{so}_6)$

$\mathcal{U}(\mathfrak{so}_6)$ can be used to represent a symmetric particle system. However, allowing a parameter $0 < q \leq 1$ allows drift and an asymmetric particle system.
Thus, we use the q-deformed quantum group, $\mathcal{U}_q(\mathfrak{so}_6)$, which was created by both [Dri85] and [Jim85] independently.

# Relations of $\mathcal{U}_q(\mathfrak{so}_6)$

$\mathcal{U}_q(\mathfrak{so}_6)$ is generated by $\{E_1, E_2, E_3, F_1, F_2, F_3, q^{H_1}, q^{H_2}, q^{H_3}\}$ with relations:

$$[E_i, F_i] = \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}};$$

$$q^{H_i} E_j = q^{\alpha_i \cdot \alpha_j} E_j q^{H_i};$$

$$q^{H_i} F_j = q^{-\alpha_i \cdot \alpha_j} F_j q^{H_i}$$

for $1 \leq i, j \leq 3$ and

$$E_l^2 E_k + E_k E_l^2 = (q + q^{-1}) E_l E_k E_l; \quad F_l^2 F_k + F_k F_l^2 = (q + q^{-1}) F_l F_k F_l$$

for $(l, k) \in \{(1, 2), (2, 1), (1, 3), (3, 1)\}$ .

# Coproduct of $\mathcal{U}_q(\mathfrak{so}_6)$

We also define coproducts of the generators. This, together with the relations, makes $\mathcal{U}_q(\mathfrak{so}_6)$ a bialgebra.

$$\Delta(E_i) = E_i \otimes 1 + q^{H_i} \otimes E_i$$
$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes q^{-H_i}$$
$$\Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i}$$

# Representation of $\mathcal{U}_q(\mathfrak{so}_6)$

$\mathcal{U}_q(\mathfrak{so}_6)$ can be represented as a subset of $M_{6\times 6}(\mathbb{R}[q, q^{-1}])$ as follows:

| | $E_i$ | $F_i$ | $q^{H_i}$ |
|---|---|---|---|
| i=1 | $E_{1,2} - E_{5,4}$ | $E_{2,1} - E_{4,5}$ | $qE_{1,1} + q^{-1}E_{2,2} + E_{3,3} + q^{-1}E_{4,4} + qE_{5,5} + E_{6,6}$ |
| i=2 | $E_{2,3} - E_{6,5}$ | $E_{3,2} - E_{5,6}$ | $E_{1,1} + qE_{2,2} + q^{-1}E_{3,3} + E_{4,4} + q^{-1}E_{5,5} + qE_{6,6}$ |
| i=3 | $E_{3,5} - E_{2,6}$ | $E_{5,3} - E_{6,2}$ | $E_{1,1} + qE_{2,2} + qE_{3,3} + E_{4,4} + q^{-1}E_{5,5} + q^{-1}E_{6,6}$ |

# REU 2020's Central Element (page 1)

$$
\begin{aligned}
q^{-4-2H_1-H_2-H_3} + q^{-2-H_2-H_3} + q^{H_2-H_3} + q^{H_3-H_2} + q^{2+H_2+H_3} \\
+q^{4+2H_1+H_2+H_3} + \frac{r^2}{q^3}F_1 q^{-H_1-H_2-H_3}E_1 + \frac{r^2}{q}F_2 q^{-H_3}E_2 \\
+\frac{r^2}{q}F_3 q^{-H_2}E_3 + r^2 q F_2 q^{H_3}E_2 + r^2 q F_3 q^{H_2}E_3 + r^2 q^3 F_1 q^{H_1+H_2+H_3}E_1 \\
+\frac{r^2}{q^3}(qF_{12} - F_{21})q^{-H_1-H_3}(qE_{21} - E_{12}) \\
+\frac{r^2}{q^3}(qF_{13} - F_{31})q^{-H_1-H_2}(qE_{31} - E_{13})
\end{aligned}
$$

# REU 2020's Central Element (page 2)

$$
\begin{aligned}
&+ r^2 q(qF_{21} - F_{12})q^{H_1+H_3}(qE_{12} - E_{21}) \\
&- r^2 q(qF_{31} - F_{13})q^{H_1+H_2}(qE_{13} - E_{31}) \\
&- \frac{r^2}{q^3}(q^2 F_{123} - qF_{213} - qF_{312} + F_{231})q^{-H_1}(q^2 E_{231} - qE_{312} - qE_{213} + E_{123}) \\
&- \frac{r^2}{q}(q^2 F_{231} - qF_{312} - qF_{213} + F_{123})q^{H_1}(q^2 E_{123} - qE_{213} - qE_{312} + E_{231}) \\
&- \frac{r^4}{q^2}((q^2+1)F_{1231} - qF_{1312} - qF_{2131})((q^2+1)E_{1231} - qE_{1312} - qE_{2131}) \\
&- r^4 F_2 F_3 E_2 E_3
\end{aligned}
$$

[Kua+20]

# The Algebraic Approach

Our goals:

- Use the 2020 REU's central element $C$ to algebraically compute a generator for a reversible Markov process
- Compare results with the probabilistic approach

# Algebraic Steps

How do we get a Markov process generator algebraically?

1. Compute the 20-dimensional irreducible representation $W$ of $\mathcal{U}_q(\mathfrak{so}_6)$
2. Compute $\pi_{W\otimes W}(C)$, the $400 \times 400$ matrix corresponding to REU 2020's central element in the representation $W \otimes W$
3. Decompose $W \otimes W$ into a direct sum of weight spaces to block $\pi_{W\otimes W}(C)$
4. Apply the method of [Kua19] to produce a ground state transformation of $\pi_{W\otimes W}(C)$
5. Apply this ground state transformation to produce a matrix whose rows sum to 0

# Weight Modules

We define $P$ to be the weight lattice of $\mathcal{U}_q(\mathfrak{so}_6)$, with $\lambda \in P$ called a weight.

$v_\lambda$ is called a highest weight vector if $E_1 v_\lambda = E_2 v_\lambda = E_3 v_\lambda = 0$.

Then, a $\mathcal{U}_q(\mathfrak{so}_6)$-module $M$ is a highest weight module if

$$M = \{Av_\lambda | A \in \mathcal{U}_q(\mathfrak{so}_6)\}.$$

# Highest Weight Modules

Since $\mathcal{U}_q(\mathfrak{so}_6)$ can be triangularly decomposed into its three subalgebras generated by $\{E_i\}$, $\{q^{H_i}\}$, and $\{F_i\}$.

$M$ can be computed by multiplying the representations of $F_i$s with a highest weight vector $v_\lambda$.

$\{\pi_V(E_i)\}$ annihilates $e_1 \otimes e_1$. We generate its corresponding $\mathcal{U}_q(\mathfrak{so}_6)$-module $W$ next.

# Finding $W$

$W$ is the vector subspace of $\mathbb{R}^6 \otimes \mathbb{R}^6$ spanned by vectors of one of the following forms, where $1 \leq i, j, k, \ell, m \leq 3$:

- $e_1 \otimes e_1$
- $\pi_V(F_i)(e_1 \otimes e_1)$
- $\pi_V(F_iF_j)(e_1 \otimes e_1)$
- $\pi_V(F_iF_jF_k)(e_1 \otimes e_1)$
- $\pi_V(F_iF_jF_kF_\ell)(e_1 \otimes e_1)$
- $\pi_V(F_iF_jF_kF_\ell F_m)(e_1 \otimes e_1)$

# $W$ is Irreducible

A finite-dimensional highest weight module is irreducible. Thus, by construction, $W$ is an irreducible representation. Thus, $\pi_W(C)$ is actually a diagonal matrix. We confirmed computationally that

$$\pi_W(C) = \left(q^8 + q^2 + 2 + q^{-2} + q^{-8}\right) \mathrm{Id}_{20}$$

# Motivation: Crystal Bases

In the irreducible representation of $M_{6\times 6}(\mathbb{R}[q, q^{-1}])$, we had basis vectors $e_1, \ldots, e_6$.
However, in the representation $V$, our basis vectors were $q$-deformed (which matches its decomposition into irreducible representations).
By letting $q \to 0$, the $q$-deformed basis vectors match the basis vectors $e_i \otimes e_j$.

# Motivation: Crystal Bases, cont

This motivates crystal bases - by letting $q \to 0$, we can get a simpler expression for the bases of $\mathcal{U}_q(\mathfrak{so}_6)$-modules.
Doing this will allow us to decompose tensors of irreducible representations (namely, $W \otimes W$) into a direct sum of irreducible representations. From there, we can decompose each irreducible representation into weight spaces. Then, from the weight spaces we can recover the eigenvalues and multiplicities of each block in the 400x400 Hamiltonian then compare them to those of the probability group.
Background and theorems on crystal bases are from [HK02].

# Roots and Weights

Let $L_i$ be the linear functional taking a matrix to its $i^{th}$ diagonal entry.

Then, $\mathfrak{so}_6$ has simple roots

- $\alpha_1 = L_1 - L_2$
- $\alpha_2 = L_2 - L_2$
- $\alpha_3 = L_2 + L_3$

and fundamental weights

- $\omega_1 = L_1$
- $\omega_2 = \frac{1}{2}(L_1 + L_2 - L_3)$
- $\omega_3 = \frac{1}{2}(L_1 + L_2 + L_3)$.

# $q$-integers $\mathbb{C}(q)$

Define a $q$-integer to be of the form

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$$

which are elements of the field $\mathbb{C}(q)$. Then, define

$$F_i^{(k)} u = \frac{1}{[k]_q!} F_i^k u$$

# Kashiwara Operators

The Kashiwara operators permute tensor products of basis vectors in a natural way. Rigorously, given a decomposition of a weight vector

$$u = \sum_{k=0}^{n} F_i^{(k)} u_k$$

for each $i = 1, 2, 3$, we define

$$\tilde{E}_i = \sum_{k=1}^{N} F_i^{(k-1)} u_k$$

and

$$\tilde{F}_i = \sum_{k=0}^{N} F_i^{(k+1)} u_k$$

# $\mathcal{L}(\lambda), \mathcal{B}(\lambda)$

$W$ is the irreducible highest weight $\mathcal{U}_q(\mathfrak{so}_6)$-module with highest weight $\lambda$ and highest weight vector $v_\lambda$. Define:

- $\mathcal{L}(\lambda)$ to be a free submodule of $W$ spanned by $\{\tilde{F}_{i_1}...\tilde{F}_{i_n}e_1 \otimes e_1\}$ with each $i_k \in \{1, 2, 3\}$ and $n \geq 0$. This is the crystal lattice of $W$.
- $\mathcal{B}(\lambda)$ to be

$$\{\tilde{F}_{i_1}...\tilde{F}_{i_n}e_1 \otimes e_1 + q\mathcal{L}(\lambda) | i_k \in \{1, 2, 3\}, n \geq 0\} \setminus \{0\}$$

# Crystal Limit

A crystal lattice $\mathcal{L}$ is a submodule over the localization of $\mathbb{C}[q]$, and is a principal ideal domain. Thus, modding by its unique maximal ideal $q\mathcal{L}$, we obtain the equivalent of letting $q \to 0$. Projecting $v \in \mathcal{L}$ to $\bar{v} \in \mathcal{L}/q\mathcal{L}$ is known as taking the crystal limit of $v$.

# Crystal Bases, defn

A pair $(\mathcal{L}, \mathcal{B})$ is a crystal base of a $\mathcal{U}_q(\mathfrak{so}_6)$-module $M$ if it satisfies the following conditions:

- $\mathcal{L}$ is a crystal lattice of $M$
- $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L}$
- $\mathcal{B} = \bigsqcup \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap \mathcal{L}/q\mathcal{L}$, $\lambda \in P$
- $\tilde{E}_i\mathcal{B} \subset \mathcal{B} \cup \{0\}$, $\tilde{F}_i\mathcal{B} \subset \mathcal{B} \cup \{0\}$ for each $i = 1, 2, 3$
- $\tilde{F}_i b_1 = b_2$ iff $b_1 = \tilde{E}_i b_2$ for every $b_1, b_2 \in \mathcal{B}$, $i = 1, 2, 3$

# Crystal Graphs

An arrow superscripted with $i$ from $v_j$ to $v_k$ symbolizes that $\tilde{F}_i v_j = v_k$. $\boxed{j}$ represents $v_j$, and $\boxed{\bar{j}}$ represents $v_{\bar{j}}$.

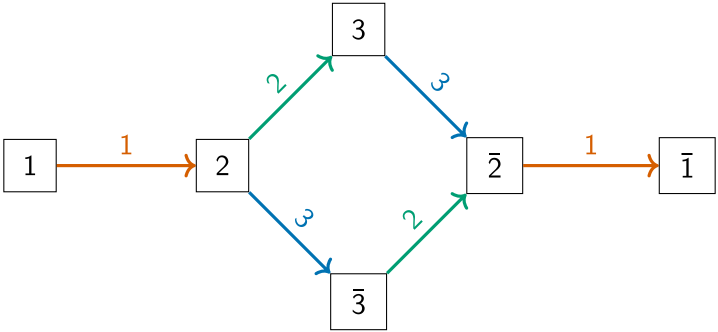



Figure: Crystal Graph of $V$

# Tensor Product Theorem

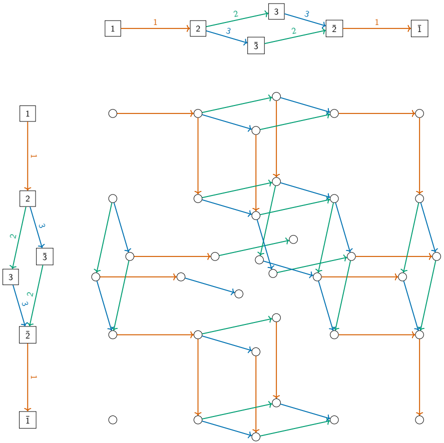
Let $V(\lambda)$, $V(\mu)$ be $\mathcal{U}_q(\mathfrak{so}_6)$-modules with corresponding crystal bases $(\mathcal{L}(\gamma), \mathcal{B}(\gamma))$, $(\mathcal{L}(\mu), \mathcal{B}(\mu))$.

Then, $(\mathcal{L}(\gamma) \otimes \mathcal{L}(\mu), \mathcal{B}(\gamma) \times \mathcal{B}(\mu))$ is a crystal basis of $V(\lambda) \otimes V(\mu)$, where the action of $\tilde{F}_i$ is defined by:

$$\tilde{F}_i(b_1 \otimes b_2) = \begin{cases} \tilde{F}_i b_1 \otimes b_2 & \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{F}_i b_2 & \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

More explicit information can be found in [HK02].

# Crystal Graph of $V \otimes V$




$$\mathcal{B}(\omega_1) \otimes \mathcal{B}(\omega_1) = \mathcal{B}(2\omega_1) \oplus \mathcal{B}(L_1 + L_2) \oplus \mathcal{B}(0)$$

# Crystal Graph of $W$

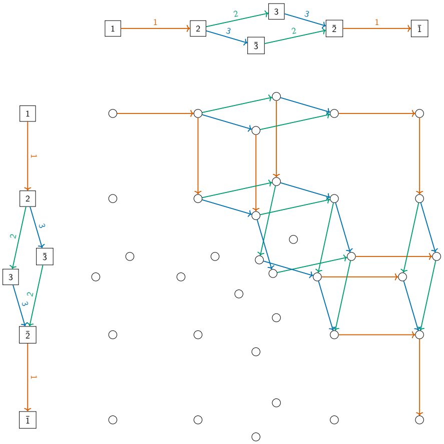



Figure: Crystal graph of $V(2L_1)$ in the representation $V \otimes V$

# Decomposing $W \otimes W$

$$\mathcal{B}(\mathcal{Y}[v_1, v_1]) = \mathcal{B}\left(\ydiagram{4}\right)$$

$$\mathcal{B}(\mathcal{Y}[v_2, v_1]) = \mathcal{B}\left(\ydiagram{3,1}\right)$$

$$\mathcal{B}(\mathcal{Y}[v_2, v_2]) = \mathcal{B}\left(\ydiagram{2,2}\right)$$

$$\mathcal{B}(\mathcal{Y}[v_{\bar{1}}, v_1]) = \mathcal{B}\left(\ydiagram{2}\right)$$

$$\mathcal{B}(\mathcal{Y}[v_{\bar{1}}, v_2]) = \mathcal{B}\left(\ydiagram{1,1}\right)$$

$$\mathcal{B}(\mathcal{Y}[v_{\bar{1}}, v_{\bar{1}}]) = \mathcal{B}(\varnothing).$$

# Decomposing $W \otimes W$, cont

$$\mathcal{B}(\mathcal{Y}) \otimes \mathcal{B}(\mathcal{Y}) \cong \mathcal{B}\left(\ydiagram{4}\right) \oplus \mathcal{B}\left(\ydiagram{3,1}\right) \oplus \mathcal{B}\left(\ydiagram{2,2}\right) \oplus \mathcal{B}\left(\ydiagram{2}\right) \oplus \mathcal{B}\left(\ydiagram{1,1}\right) \oplus \mathcal{B}(\varnothing)$$

# Decomposing $W \otimes W$, cont

$$W \otimes W \cong V(4L_1) \oplus V(3L_1 + L_2) \oplus V(2L_1 + 2L_2) \oplus V(2L_1) \oplus V(L_1 + L_2) \oplus V(0)$$

The tensor product rule for Young tableaux can be found explicitly in [HK02].

# Weight Space Decomposition of $W \otimes W$

$W \otimes W$ can be expressed as a direct sum of weight spaces:

$$W \otimes W = \bigoplus_{i,j,k} W \otimes W[i,j,k]$$

Where $(i,j,k)$ range over all $(i,j,k)$ with $|i| + |j| + |k| = 0$, 2, or 4.

Any matrix in $W \otimes W$ (as a representation of $\mathcal{U}_q(\mathfrak{so}_6)$) can be written as a direct sum of 85 blocks with sizes equal to the dimension of the corresponding weight space.

# Weight Space Dimensions of $V$

The dimension of a weight space can also be determined by crystal graphs.

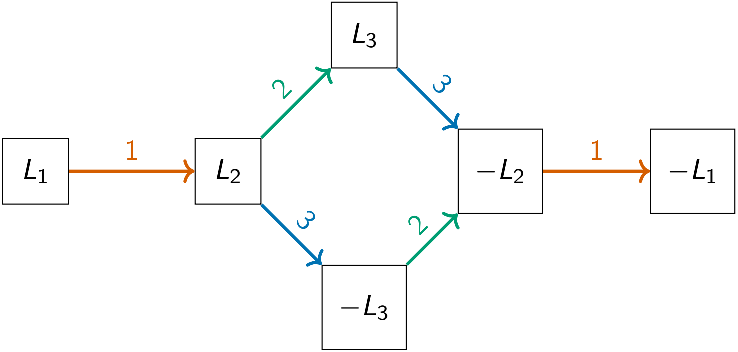



Figure: Weights in the Crystal Graph of $V$

# Weight Space Dimensions of Tensor Products

Let $\mathfrak{g}$ be a Lie algebra, and $V_1$ and $V_2$ be two representations of $\mathcal{U}_q(\mathfrak{g})$. Then for any two basis vectors $v_1 \in V_1$, $v_2 \in V_2$ with respective weights $\lambda_1$ and $\lambda_2$, the basis vector $v_1 \otimes v_2 \in V_1 \otimes V_2$ has weight $\lambda_1 + \lambda_2$. Thus, we can use crystal graphs to determine the weights of $W$ from the weights of $V$, and the weights of $W \otimes W$ from the weights of $W$.

# Weight Spaces of $W$

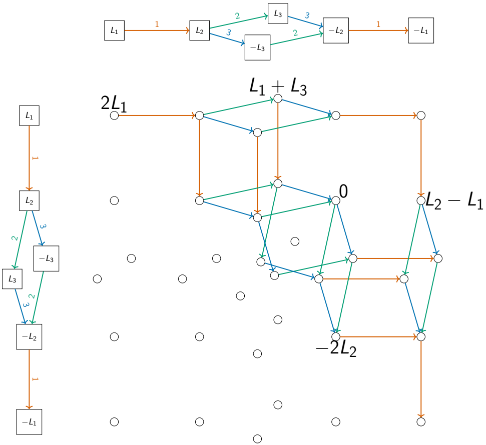



Figure: Weights in the crystal graph of $W$

# Weight Spaces of $W \otimes W$

Every weight space of $W \otimes W$ has a weight taking one of the following forms:

- $0$
- $L_i \pm L_j$
- $2L_i$
- $2L_i \pm L_j \pm L_k$
- $2L_i \pm 2L_j$
- $3L_i \pm L_j$
- $4L_i$

# Example: Dimension of weight space with weight $2L_1 + L_2 + L_3$

We count the number of pairs $(w_1, w_2)$ of basis vectors in the crystal graph of $W$ such that the sum of their respective weights $\lambda_1$ and $\lambda_2$ takes the form

$$\lambda_1 + \lambda_2 = 2L_1 + L_2 + L_3$$

By inspecting the weights of $W$, we can conclude that there are exactly four possible pairs $(w_1, w_2)$:

1. $w_1 = e_1 \otimes e_2$, $w_2 = e_1 \otimes e_3$. Then $\lambda_1 = L_1 + L_2$, $\lambda_2 = L_1 + L_3$.
2. $w_1 = e_1 \otimes e_3$, $w_2 = e_1 \otimes e_2$. Then $\lambda_1 = L_1 + L_3$, $\lambda_2 = L_1 + L_2$.
3. $w_1 = e_1 \otimes e_1$, $w_2 = e_2 \otimes e_3$. Then $\lambda_1 = 2L_1$, $\lambda_2 = L_2 + L_3$.
4. $w_1 = e_2 \otimes e_3$, $w_2 = e_1 \otimes e_1$. Then $\lambda_1 = L_2 + L_3$, $\lambda_2 = 2L_1$.

Note that $e_2 \otimes e_1$, $e_3 \otimes e_1$, and $e_3 \otimes e_2$ are not in the crystal graph of $W$, and thus are not valid possibilities for $w_1 \otimes w_2$.

# Blocking $\pi_{W\otimes W}(C)$

Similar combinatorial arguments for the remaining types of weight spaces yield the following block decomposition for any element of $\mathcal{U}_q(\mathfrak{so}_6)$ represented in $W \otimes W$. In particular, $\pi_{W\otimes W}(C)$ can be written as a direct sum of

- one $22 \times 22$ block
- twelve $12 \times 12$ blocks
- six $8 \times 8$ blocks
- twenty-four $4 \times 4$ blocks
- twelve $3 \times 3$ blocks
- twenty-four $2 \times 2$ blocks
- and six $1 \times 1$ blocks.

# Ground State Transformation

A **ground state transformation** of a Hamiltonian matrix $H$ is a transformation yielding a matrix whose rums sum to 0:

$$H \mapsto a^{-1}(G^{-1}HG) - \mathrm{Id}$$

Where $G$ is a diagonal matrix and $a$ is the eigenvalue of $H$ associated with the highest weight vector of the irreducible representation $H$ lives in.

Since the matrix $a^{-1}(G^{-1}HG) - \mathrm{Id}$ has rows summing to 0, it generates a Markov process.

# Generating $G$

The following is true for type $A$ Lie algebras:

For any basis vector $e_i \otimes e_j \in W \otimes W$, there exists a unique product $P = F_1^{\eta_1} F_2^{\eta_2} \cdots F_k^{\eta_k}$ of generators $F_1, F_2, \ldots, F_k$ in any order such that

$$(e_i \otimes e_j)^T P(e_1 \otimes e_1) \neq 0$$

If we try to extend this to our specific case of $\mathfrak{so}_6$ (which, notably, is a type $D$ Lie algebra), we have $k = 3$.

# Applying the Type A method

The Type A method only works for some basis vectors of $W \otimes W$. When it does work, we set the corresponding diagonal entry of $G$ to the nonzero value produced:

$$(e_i \otimes e_j)^T P(e_1 \otimes e_1) \neq 0$$

$$g_{20i+j-20} := (e_i \otimes e_j)^T P(e_1 \otimes e_1)$$

# Transforming States Groundly: Eigenvalue Equations 1

$$\begin{aligned} a^{-1}(G^{-1}HG) - \mathrm{Id} &= L \\ G^{-1}HG - a\mathrm{Id} &= aL \\ G^{-1}HG &= aL + a\mathrm{Id} \\ HG &= aGL + aG \end{aligned}$$

Since the rows of $L$ sum to 0 and $G$ is diagonal, the rows of $GL$ sum to 0 as well. Therefore, the equation above implies that the sum of each row of $HG$ equals the sum of the corresponding row of $aG$.

# Transforming States Groundly: Eigenvalue Equations 2

Let $g_i$ denote the $i$th diagonal entry of $G$. Then $G$ takes the form

$$G = \begin{bmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_{400} \end{bmatrix}$$

Thus, we can express the sum of the elements in the $i$th row of $HG$ as $\sum_{j=1}^{400} H_{i,j} g_j$, and the sum of the elements in the $i$th row of $aG$ as $ag_i$ (since $G$ is diagonal, there is one nonzero term in each row).

# Transforming State Groundly: Eigenvalue Equations 3

$$\sum_{j=1}^{400} H_{1,j} g_j = a g_1 \qquad \cdots \qquad \sum_{j=1}^{400} H_{400,j} g_j = a g_{400}$$

Thus, the vector

$$\vec{g} := \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{400} \end{pmatrix}$$

satisfies $H\vec{g} = a\vec{g}$.

# Partial Type $A$ Ground State Transformation

The process of finding $G$ (as defined for type A Lie algebras) uniquely defines a partial ground state transformation of $H = \pi_{W \otimes W}(C)$. Solving the system of eigenvalue equations described above produces a solution space with 37 unknowns.

# Extending to a Ground State Transformation

Setting each of these unknowns to zero and discarding rows/columns in the resulting matrix $a^{-1}G^{-1}\pi_{W\otimes W}(C)G - \mathrm{Id}$ which are all zero or have negative off-diagonal entries results in a Markov process generator.

This matrix is a direct sum of blocks which have a similar structure as the probability group's Markov process generator, but different entries.

# Difference with Probability

A ground state transformation of $\pi_{W \otimes W}(C)$ could never result in a $4 \times 4$ block matching the block corresponding to the communicating class

$$\big(\langle 3, 0\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 0, 3\rangle\big)$$

in the probabilistically-generated Markov generator.

# References I

[Bli64] Robert Blinc. *Relaxation phenomena and realted topics*. 1964.

[Bly+23] Danyil Blyschak et al. "Orthogonal Polynomial Duality of a Two-Species Asymmetric Exclusion Process". In: *Journal of Statistical Physics* 190.5 (May 2023). ISSN: 1572-9613. DOI: 10.1007/s10955-023-03100-y. URL: http://dx.doi.org/10.1007/s10955-023-03100-y.

# References II

[DF90] Persi Diaconis and James Allen Fill. "Strong Stationary Times Via a New Form of Duality". In: *The Annals of Probability* 18.4 (1990), pp. 1483–1522. DOI: 10.1214/aop/1176990628. URL: https://doi.org/10.1214/aop/1176990628.

[Dri85] V. G. Drinfeld. "Hopf algebras and the quantum Yang–Baxter equation". In: *Dokl. Akad. Nauk SSSR* 283 (1985), pp. 1060–1064.

[Fer] Renato Feres. "Notes for Math 450 Continuous-time Markov chains and Stochastic Simulation".

# References III

[HK02] Jin Hong and Seok-Jin Kang. *Introduction to Quantum Groups and Crystal Bases*. Providence: American Mathematical Society, 2002.

[Jim85] Michio Jimbo. "A q difference analog of U(g) and the Yang-Baxter equation". In: *Lett. Math. Phys.* 10 (1985), pp. 63–69. DOI: 10.1007/BF00704588.

[Kua] Jeffrey Kuan. *Markov Chains and Processes on Discrete State Spaces*.

[Kua+20] Jeffrey Kuan et al. *Interacting particle systems with type D symmetry and duality*. 2020. arXiv: 2011.13473.

# References IV

[Kua19] Jeffrey Kuan. "Stochastic Fusion of Interacting Particle Systems and Duality Functions". In: (2019). arXiv: 1908.02359.

[Mar06] A.A. Markov. "Extension of the law of large numbers to quantities, depending on each other (1906). Reprint.". eng. In: *Journal Électronique d'Histoire des Probabilités et de la Statistique [electronic only]* 2.1b (2006), Article 10, 12 p., electronic only–Article 10, 12 p., electronic only. URL: http://eudml.org/doc/128778.

# References V

[RLY23] Eddie Rohr, Karthik Sellakumaran Latha, and Amanda Yin. *A Type D Asymmetric Simple Exclusion Process Generated by an Explicit Central Element of* $\mathcal{U}_q(\mathfrak{so}_{10})$. 2023. arXiv: 2307.15660.

[Spi70] Frank Spitzer. "Interaction of Markov processes". In: *Advances in Mathematics* 5.2 (1970), pp. 246–290. ISSN: 0001-8708. DOI: https://doi.org/10.1016/0001-8708(70)90034-4. URL: https://www.sciencedirect.com/science/article/pii/0001870870900344.