Chamber Depths of Discriminant Contours with Four Cusps

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July 22, 2024

Introduction

Using properties of the signed reduced $\mathscr{A}$-discriminant $\Gamma_\varepsilon(\mathscr{A}, B)$ in conjunction with Morse Theory, Deng and Rojas are working to prove that the real zero set of n-variate exponential sums with $(n+3)$ terms and full-dimensional signed support $(\mathscr{A}, \varepsilon)$ have no more than three connected components.

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I will be focusing on one claim that has yet to be proven: If $\Gamma_\varepsilon(\mathscr{A}, B)$ has exactly 4 cusps, then the depth of the chambers is at most 2. Moreover, there is at most one inner chamber with depth 2.

Definition

Recall that a support $\mathscr{A} := \{\alpha_1, \cdots, \alpha_{n+k}\} \subset \mathbb{R}^n$ a set of vectors which act as exponent vectors in the exponential sum

$$f_c = \sum_{i=1}^{n+k} c_i e^{x \cdot \alpha_i}$$

along with $\varepsilon := \text{sign}\,(c_i) \in \{\pm 1\}^{n+k}$ is called the *signed support* of f_c.

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Definition

Given a signed support $(\mathscr{A}, \varepsilon)$, define the *signed $\mathscr{A}$-discriminant* to be

$$\nabla_{\mathscr{A},\varepsilon} = \{c \in \mathbb{R}^{n+k} | Sign(c) = \varepsilon \ \& \ Sing(f_c) \neq \emptyset\}$$

Definitions

Definition

For a fixed signed support $(\mathscr{A}, \varepsilon)$, let $B \in \mathbb{R}^{(n+k)\times(k-1)}$ be the Gale dual matrix of $\mathscr{A}$, a matrix of maximal rank whose columns are a basis of the kernel of $\hat{\mathscr{A}} = \begin{bmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_m \end{bmatrix} \in \mathbb{R}^{(n+1)\times m}$.

We define the *signed reduced $\mathscr{A}$-discriminant* $\Gamma_\varepsilon(\mathscr{A}, B)$ to be

$$\Gamma_\varepsilon(\mathscr{A}, B) = B^\top Log|\nabla_{\mathscr{A},\varepsilon}|$$

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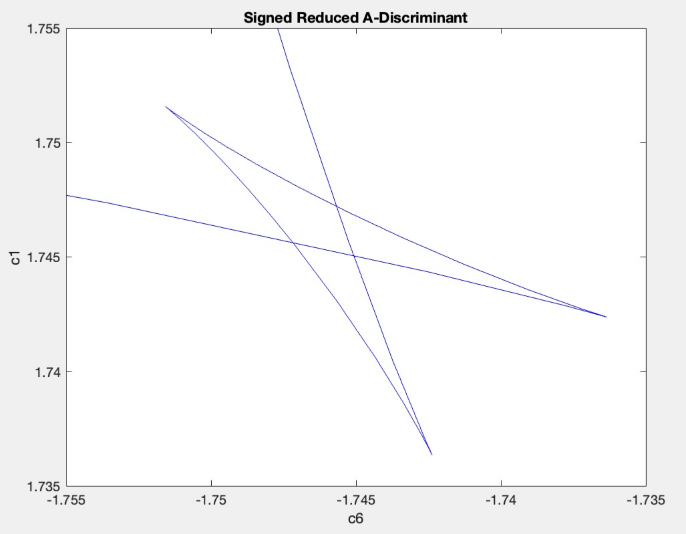
Proposition

Let $(\mathscr{A}, \varepsilon)$ be a full-dimensional signed support with Gale dual matrix B. The image of the map
$\xi_{B,\varepsilon} : \{\lambda \in \mathbb{R}^{k-1} | sign(B\lambda) = \varepsilon\} \to \mathbb{R}^{k-1}, \quad \xi_{B,\varepsilon}(\lambda) = B^\top Log|B\lambda|$
is the signed reduced $\mathscr{A}$-discriminant $\Gamma_\varepsilon(\mathscr{A}, B)$.

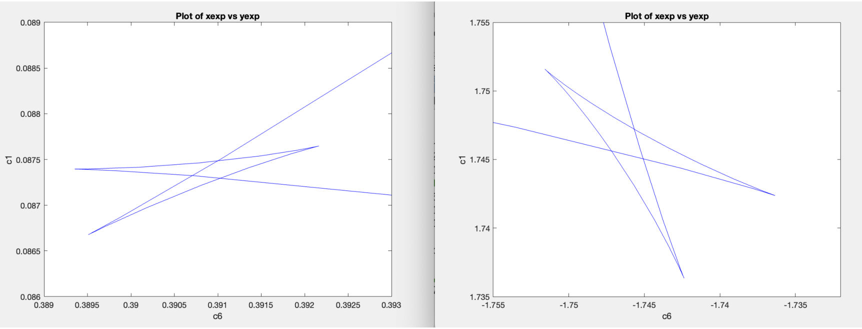
Signed Reduced $\mathscr{A}$-Discriminant

Example



Choice of Gale Dual Matrix

Here are two discriminant contours $\Gamma_{\varepsilon}(\mathscr{A}, B_1)$ and $\Gamma_{\varepsilon}(\mathscr{A}, B_2)$ which correspond to the same signed support.



Chambers

Definition

Call connected components of $\mathbb{R}^{k-1} \setminus \Gamma_\varepsilon(\mathscr{A}, B)$ signed reduced chambers. The bounded chambers are inner chambers and unbounded chambers are outer chambers.

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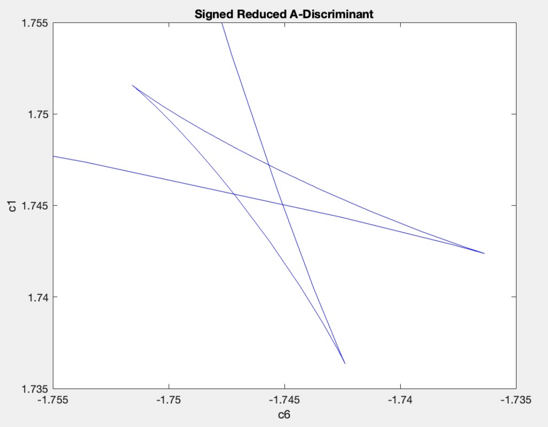

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Proposition

Let $(\mathscr{A}, \varepsilon)$ be a full-dimensional signed support with Gale dual matrix B and let $c, c' \in \mathbb{R}^{n+k}_\varepsilon$. If $B^\top Log|c|$ and $B^\top Log|c'|$ are in the same connected component of

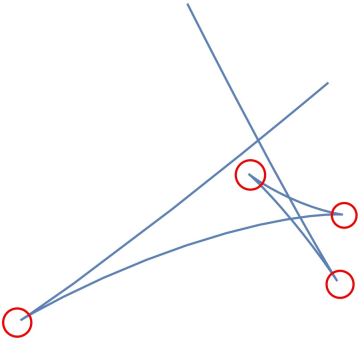
$$\mathbb{R}^{k-1} \setminus (\bigcup_{F \subset Conv(\mathscr{A}) a face} B^\top Log|\tilde{\nabla}_{\mathscr{A}_F, \varepsilon_F}),$$

then the zero sets $Z_\mathbb{R}(f_c)$ and $Z_\mathbb{R}(f_{c'})$ are ambiently isotopic in $\mathbb{R}^n$.

Chamber Depth

Definition

For a given chamber, consider the possible paths from a point inside this chamber to a point in an outer chamber. The minimum number of intersections of this path with $\Gamma_\varepsilon(\mathscr{A}, B)$ is defined to be the depth of that chamber.



[Dent, Rojas, Telek, 24]

Chamber Depth and Isotopy Type

Lemma

Suppose f_c *is an honest* n*-variate exponential sum with* $n+3$ *terms which corresponds to a point in the coefficient space in an outer chamber. Then* $Z_{\mathbb{R}}(f_c)$ *has at most* 2 *connected components.*

[Bihan, Humbert, Tavenas, 2022]

Lemma

When the coefficient vector of an exponential sum f *crosses* $\Gamma_{\varepsilon}(\mathscr{A}, B)$*, the number of connected components of* $Z_{\mathbb{R}}(f)$ *changes by* $-1, 0,$ *or* 1*. [Katz73]*

Lemma

If f_c *corresponds to a point* c *in a chamber of depth* d*, then* $Z_{\mathbb{R}}(f_c)$ *has at most* $2+d$ *connected components.*

Properties of the Signed Reduced $\mathscr{A}$-Discriminant

Definition

The contour $\Gamma_\varepsilon(\mathscr{A}, B)$ is piece-wise smooth between its critical points, which are called cusps. Call each smooth part a curve segment which contains its cusp(s).

Lemma

For the parameterization $\xi_{B,\varepsilon}$ of the signed reduced $\mathscr{A}$-discriminant given by

$$\xi_{B,\varepsilon}(\lambda) = B^\top Log|B\lambda|,$$

for each $\lambda \in \{\lambda \in \mathbb{R}^{k-1} | sign(B\lambda) = \varepsilon\}$, the vector normal to $\Gamma_\varepsilon(\mathscr{A}, B)$ at $\xi_{B,\varepsilon}(\lambda)$ is λ.

[Kapranov, 2000]

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Lemma

Every pair of curve segments intersect at most once.

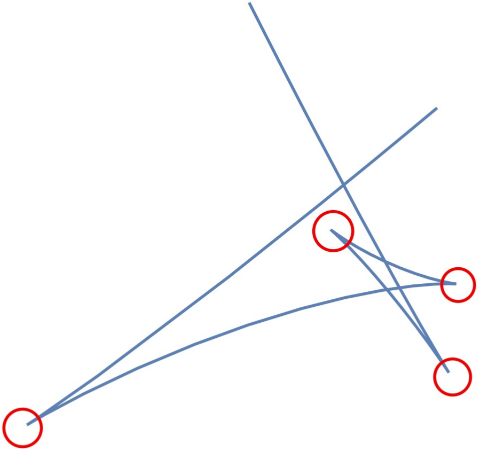
Reducing Chambers to Polyhedron

We reduce $\mathbb{R}^{k-1} \setminus \Gamma_\varepsilon(\mathscr{A}, B)$ and its chamber structure to a "straightened out" version by replacing curve segments with straight lines connecting cusps.
This turns inner chambers into polyhedron. In our case of (n+3)-nomials, $\Gamma_\varepsilon(\mathscr{A}, B) \subset \mathbb{R}^2$ so our chambers become polygons. Now we can apply Euler's characteristic $V - E + F = 2$.

Lemma

Suppose $\Gamma_\varepsilon(\mathscr{A}, B)$ *has* m *cusps with* $m \geq 2$. *Then* $\mathbb{R}^{k-1} \setminus \Gamma_\varepsilon(\mathscr{A}, B)$ *has 2 outer chambers and at most* $\frac{m(m-1)}{2}$ *inner chambers.*

The Problem at Hand



Dent, Rojas, Telek, 24

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If there exists a chamber of depth 3, then there are at least two chambers of depth at least 2 adjacent to it. If we can prove the latter part of the claim, that there is at most one chamber with depth 2, then the proof is complete.

Bounds on Intersections, Vertices, and Edges in the 4 Cusp Case

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These two lemmas allow us to bound the number of inner chambers by 6, the number of intersections (including cusps) by 10, and the number of distinct edges to chambers by 17.