Solutions to AB Exam

Texas A&M High School Math Contest
2 November, 2024

1. All positive integers are written consecutively (starting from 1) as a single sequence of decimal digits. Find the 2024th digit in that sequence.

   Solution: The 9 one-digit numbers fill the first 9 digits in the sequence. The 90 two-digit numbers fill the next $2 \cdot 90 = 180$ digits. The 900 three-digit numbers fill the next $3 \cdot 900 = 2700$ digits. Since $2024 = 9 + 180 + 3 \cdot 611 + 2$, the 2024th digit is the 2nd digit of the 612th three-digit number. The 1st three-digit number is 100. Hence the 612th three-digit number is 711. Its 2nd digit is 1.

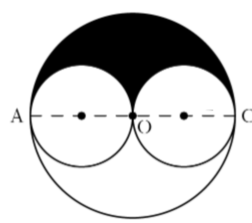
   Answer: 1.

2. If you walk for 45 minutes at a rate of 4 mph and then run for 30 minutes at a rate of 10mph, what is your average speed over the entire trip, in miles per hour?

   Solution: You traveled $\left(\frac{3}{4}\right)(4) = 3$ miles walking and $\left(\frac{1}{2}\right)(10) = 5$ miles running, so you traveled 8 miles in 75 minutes $\left(\frac{5}{4} \text{ hour}\right)$. Your average speed is $\frac{8}{5/4} = \frac{32}{5} = 6.4$ mph**.**

   Answer: $\frac{32}{5} = 6.4$ mph.

3. In the figure below, circle O has diameter $\overline{AC}$, and the smaller circles have diameters $\overline{AO}$ and $\overline{OC}$. If the radii of the smaller circles is 24, what is the shaded area?



   Solution: The shaded area is one-half the area of circle O minus the area of one smaller circle.

$$A = \frac{1}{2}\pi(48)^2 - \pi(24)^2$$

$$A = 24\pi(48 - 24) = \pi(24)^2 = 576\pi$$

   Answer: 576π square units

4. In a soccer tournament, every two teams played each other twice. What was the number of participating teams if the total number of games played was 182?

   Solution: Without loss of generality we may assume that in every pair of teams, each team hosted the other one once and visited the other one once. Then every team hosted every other team exactly once. It follows that the number of games played was $n(n-1)$, where n is the number of participating teams. We are given that $n(n-1)=182$. Hence $n^2-n-182=0$. This quadratic equation in n has two solutions: $n=14$ and $n=-13$. Only the positive solution makes sense here.

Answer: 14

5. The numbers -2, 11, 2, 20, and 24 are arranged according to the rules below:

   - The largest isn't first, but it is in one of the first three places.
   - The smallest isn't last, but it is in one of the last three places.
   - The median isn't first or last.

   What is the average of the first and last numbers?

   Solution: The three statements are equivalent to saying that the largest number (24), smallest number (-2), and median (11) are not first or last. Therefore, the first and last number must be 2 and 20 (in either order), so their average is 11.

   Answer: 11

6. $(2024)^2 = 4,096,576$. From this number, subtract $(2023)(2024)$, then add $(2022)(2024)$. Continue this pattern of alternating subtraction and addition all the way down to $(1)(2024)$. What is the total?

   Solution: We are asked to evaluate

   $$(2024)(2024)-(2023)(2024)+(2022)(2024)-(2021)(2024)\cdots+(2)(2024)-(1)(2024).$$

   Pair consecutive subtractions and factor:

   $$=(2024)(2024-2023)+(2024)(2022-2021)+\cdots(2024)(2-1)$$

   There are 1,012 pairs of $(2024)(1)$, so the total is

   $$(1012)(2024)=\frac{1}{2}(2024)^2=2,048,288.$$

   Answer: $2,048,288$

7. Find the minimal possible value of the expression $x+\dfrac{2}{x}$, where $x>0$.

   Solution: The expression $x+\dfrac{2}{x}$ is defined for any $x\neq 0$. Its value is positive if and only if $x>0$. Therefore our task is to find the smallest positive value of a parameter a for which the equation $x+\dfrac{2}{x}=a$ has a real solution x. The equation is equivalent to a quadratic equation $x^2-ax+2=0$. We can transform the latter as follows:

   $$x^2-ax+\frac{a^2}{4}-\frac{a^2}{4}+2=0,$$
   $$\left(x-\frac{a}{2}\right)^2-\frac{a^2}{4}+2=0,$$
   $$\left(x-\frac{a}{2}\right)^2=\frac{a^2}{4}-2.$$

It follows that a real solution x exists if and only if $a^2/4 - 2 \geq 0$ or, equivalently, $a^2 \geq 8$. The smallest positive value of a that satisfies this condition is $\sqrt{8} = 2\sqrt{2}$.

Answer: $2\sqrt{2}$

8. Solve for x: $3^x + 3^x + 3^x = 9^{2x}$.

   Solution: $3^x + 3^x + 3^x = 3(3^x) = 3^{x+1}$ and $9^{2x} = (3^2)^{2x} = 3^{4x}$, so we have

$$3^{x+1} = 3^{4x}$$

   Using a law of exponents that $a^x = a^y \Leftrightarrow x = y$, we have

$$x + 1 = 4x \Rightarrow x = \frac{1}{3}$$

   Answer: $\frac{1}{3}$

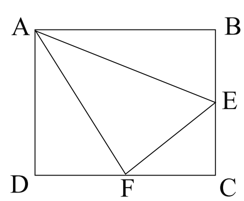
9. Find a solution (x, y, z) of the following system of equations:

$$\begin{cases} 2x + 3y + 2z = 3, \\ 3x + 2y + 2z = 5, \\ 2x + 2y + 3z = 6. \end{cases}$$

   Solution: Adding all three equations of the system, we obtain $7x + 7y + 7z = 14$. Hence $x + y + z = 2$. Then also $2x + 2y + 2z = 4$. Subtracting the latter equation from each equation of the system, we obtain that $y = -1$, $x = 1$ and $z = 2$. It remains to check that $(1, -1, 2)$ is indeed a solution of the system.

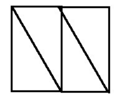
   Answer: $(1, -1, 2)$

10. Suppose the area of rectangle $ABCD$ is 224, point E is the midpoint of $\overline{BC}$, and point F is the midpoint of $\overline{CD}$ (see figure below). What is the area of $\triangle AEF$?

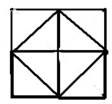


   (NOTE that since the length and width are not specified, one can choose a length and width such that the area is 224 square units and use these values in place of the generalized ℓ and w below)

   Solution: Since F is the midpoint of $\overline{DC}$, the area of $\triangle ADF$ is $\frac{1}{4}$ of the area of the rectangle. Algebraically, if ℓ and w are the dimensions of the rectangle, the area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}\left(\frac{1}{2}\ell\right)w = \frac{1}{4}\ell w$. Geometrically, the triangle covers the rectangle as shown below:



Similarly, the area of $\triangle ABE$ is $\frac{1}{4}$ of the area of the rectangle. Finally, the area of $\triangle CEF$ is $\frac{1}{8}$ of the area of the rectangle: $A = \frac{1}{2}\left(\frac{1}{2}\ell\right)\left(\frac{1}{2}w\right)$ or using the figure below



Therefore, the area of $\triangle AEF$ is $(224) - \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{8}\right)(224) = \left(\frac{3}{8}\right)(224) = (3)(28) = 84$ square units

Answer: 84 square units

11. The sum of 24 consecutive odd numbers is 12,000. What is the largest number?

Solution: Let n be the largest odd number. Then

$$n + (n-2) + (n-4) + \cdots + (n-42) + (n-44) + (n-46) = 12000$$

Pairing the largest and smallest numbers, and working our way in, we have 12 pairs of numbers which add to $2n - 46$, so

$$(2n-46)(12) = 12000 \to 2n - 46 = 1000$$

Solving this yields $n = \frac{1046}{2} = 523$.

Answer: 523

12. Find a positive number x such that $x = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}$.

Solution: For any $x > 0$ we can simplify the nested fraction step by step as follows:

$$1+\frac{1}{x} = \frac{x+1}{x}, \qquad 1+\cfrac{1}{1+\cfrac{1}{x}} = 1+\frac{x}{x+1} = \frac{2x+1}{x+1},$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}} = 1+\frac{x+1}{2x+1} = \frac{3x+2}{2x+1}.$$

Hence we are looking for a positive solution of the equation

$$x = \frac{3x+2}{2x+1} \iff \frac{x(2x+1)-(3x+2)}{2x+1} = 0 \iff \frac{2x^2-2x-2}{2x+1} = 0.$$

For a positive x, this is equivalent to the quadratic equation $x^2 - x - 1 = 0$. The quadratic equation has two solutions: $\frac{1}{2}(1-\sqrt{5})$ and $\frac{1}{2}(1+\sqrt{5})$. The solution $\frac{1}{2}(1+\sqrt{5})$ is positive.

Alternative solution: It is easy to observe that any solution of the equation $x = 1 + \dfrac{1}{x}$ is also a solution of the given equation by repeatedly substituting x for $1 + \dfrac{1}{x}$ in the right side of the given equation. The equation $x = 1 + \dfrac{1}{x}$ is equivalent to the quadratic equation $x^2 - x - 1 = 0$, which has two solutions: $\frac{1}{2}(1-\sqrt{5})$ and $\frac{1}{2}(1+\sqrt{5})$. The solution $\frac{1}{2}(1+\sqrt{5})$ is positive.

Answer: $\dfrac{1}{2}(1+\sqrt{5})$

13. A six-digit integer is written down. If the units digit is moved to the front of the number (for example, 123456 becomes 612345), the new integer is five times the original integer. What is the original integer?

    Solution: Algebraically, let x be the value of the first 5 digits of the integer and y be the units digit. The original integer is then $10x + y$, while the new integer is $100000y + x$. So

    $$100000y + x = 5(10x + y).$$

    $$49x = 99995y$$

    $$7x = 14285y$$

    So we must have $x = 14285$ and $y = 7$, making our original integer 142857.
    NOTE that this can also be solved by recognizing the pattern in the repeating decimal of fractions with denominator 7:

    $$\frac{1}{7} = .\overline{142857},\ \frac{5}{7} = .\overline{714285}.$$

    Answer: 142857

14. Ten thousand light bulbs, numbered 1 to 10000, are all initially turned on. Each bulb has a switch that toggles its state (turning it on if it's off, or off if it's on). Person 1 toggles the switch on bulb #2 (the first prime number), and every multiple of 2. Person 2 toggles the switch on bulb #3 (the second prime number) and every multiple of 3. Person 3 toggles the switch on bulb #5 (the third prime number) and every multiple of 5. The pattern continues, with Person N toggling the switch on the bulb numbered with the Nth prime number and every multiple of that prime number. After all the people have completed their turns, which of the following bulbs will be on? If none, write **NONE**:

    $$24,\ 112,\ 2024,\ 2025$$

    Solution: A bulb's switch is toggled by person N if and only if the Nth prime number is a factor of the bulb number (regardless of multiplicity). Since a switch must be toggled an even number of times to be on at the end of the process, only bulb numbers with an even number of DISTINCT prime factors will remain on:

$$24 = 2^3 \cdot 3 \text{ ON}$$
$$112 = 2^4 \cdot 7 \text{ ON}$$
$$2024 = 2^3 \cdot 11 \cdot 23 \text{ OFF}$$
$$2025 = 3^4 \cdot 5^2 \text{ ON}$$

Therefore, the correct list is 24, 112, 2025

Answer: 24, 112, 2025

15. Let c be a real solution of the equation $x^4 - 3x + 1 = 0$. Evaluate the expression $c^6 + c^4 - 3c^3 + c^2 - 3c$.

    Solution: Since $c^4 - 3c + 1 = 0$, we obtain that $c^4 = 3c - 1$. Then $c^6 = c^2(3c - 1) = 3c^3 - c^2$. It follows that

    $$c^6 + c^4 - 3c^3 + c^2 - 3c = (3c^3 - c^2) + (3c - 1) - 3c^3 + c^2 - 3c = -1.$$

    Answer: -1

16. Consider a fraction $\dfrac{6n-1}{7n+1}$, where n is a positive integer. Find the smallest value of n for which the fraction is not in lowest terms.

    Solution: The given fraction is not in lowest terms if its numerator $a = 6n - 1$ and denominator $b = 7n + 1$ have a common prime divisor p. If this is the case, then the prime number p also divides the numbers $c = b - a = n + 2$ and $d = a - 6c = -13$. Hence $p = 13$. Conversely, if the number c is divisible by 13 then so are the numbers $a = 6c - 13$ and $b = a + c$.

    Thus the given fraction is not in lowest terms if and only if the number $n + 2$ is divisible by 13. The smallest value of n for which this happens is 11.

    Answer: 11.

17. How many integer solutions (x, y) does the following system of inequalities have?

    $$\begin{cases} 2x + y < 8, \\ 3x - 4y < 1, \\ x > 0. \end{cases}$$

    Solution: The first inequality of the system is equivalent to $y < 8 - 2x$. The second inequality is equivalent to $y > \frac{3}{4}x - \frac{1}{4}$. Hence for any solution (x, y) of the system we have $\frac{3}{4}x - \frac{1}{4} < 8 - 2x$. This simplifies to $\frac{11}{4}x < \frac{33}{4}$ or $x < 3$. Besides, $x > 0$. Assuming x is an integer, we obtain that $x = 1$ or $x = 2$.

    In the case $x = 1$, the conditions on y are $\frac{3}{4} \cdot 1 - \frac{1}{4} < y < 8 - 2 \cdot 1$ or $\frac{1}{2} < y < 6$. There are five integers satisfying these conditions: 1, 2, 3, 4 and 5. In the case $x = 2$, the conditions on y are $\frac{3}{4} \cdot 2 - \frac{1}{4} < y < 8 - 2 \cdot 2$ or $\frac{5}{4} < y < 4$. These are satisfied by two integers: 2 and 3.

    Thus the system has 7 integer solutions: $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(2,2)$ and $(2,3)$.

    Answer: 7.

18. Find a real solution of the equation $\sqrt{x-1}+\sqrt{x-3}=2$.

Solution: Squaring both sides of the equation, we obtain

$$(x-1)+(x-3)+2\sqrt{(x-1)(x-3)}=4,$$

which simplifies to

$$\sqrt{(x-1)(x-3)}=4-x.$$

Squaring both sides again, we obtain $(x-1)(x-3)=(4-x)^2$, which simplifies to $4x=13$. Hence $13/4$ is the only possible solution. We check that it is indeed a solution:

$$\sqrt{\frac{13}{4}-1}+\sqrt{\frac{13}{4}-3}=\sqrt{\frac{9}{4}}+\sqrt{\frac{1}{4}}=\frac{3}{2}+\frac{1}{2}=2.$$

Answer: $\frac{13}{4}=3.25$.

19. All real solutions of the inequality $\sqrt{3-2x-x^2}>x+1$ fill an interval of the real line. Find the length of that interval.

Solution: In the case $x+1<0$, the given inequality is equivalent to

$$3-2x-x^2\geq 0 \iff (3+x)(1-x)\geq 0 \iff -3\leq x\leq 1.$$

In the case $x+1\geq 0$, the given inequality is equivalent to

$$3-2x-x^2>(x+1)^2 \iff 2x^2+4x-2<0 \iff 2(x+1)^2<4 \iff -1-\sqrt{2}<x<-1+\sqrt{2}.$$

It follows that the solution set is the union of two intervals $[-3,-1)$ and $[-1,-1+\sqrt{2})$. The union is the interval $[-3,-1+\sqrt{2})$, which has length $2+\sqrt{2}$.

Answer: $2+\sqrt{2}$.

20. How many distinct real roots does the following equation have:

$$(2x^2-5x+2)^3+(6x^2-x-1)^3=(8x^2-6x+1)^3\,?$$

Solution: Let $y(x)=2x^2-5x+2$ and $z(x)=6x^2-x-1$. Then $8x^2-6x+1=y(x)+z(x)$ so the equation can be rewritten as

$$\big(y(x)\big)^3+\big(z(x)\big)^3=\big(y(x)+z(x)\big)^3.$$

After expanding the right-hand side, we obtain

$$\big(y(x)\big)^3+\big(z(x)\big)^3=\big(y(x)\big)^3+3\big(y(x)\big)^2z(x)+3y(x)\big(z(x)\big)^2+\big(z(x)\big)^3.$$

This simplifies to

$$3\big(y(x)\big)^2z(x)+3y(x)\big(z(x)\big)^2=0,$$

which is equivalent to

$$y(x)\,z(x)\,\big(y(x)+z(x)\big)=0.$$

It follows that a real number x is a root of the given equation if and only if $y(x)=0$ or $z(x)=0$ or $y(x)+z(x)=0$. The equation $2x^2-5x+2=0$ has roots $1/2$ and 2. The equation $6x^2-x-1=0$ has roots $-1/3$ and $1/2$. The equation $8x^2-6x+1=0$ has roots $1/4$ and $1/2$. Thus the given equation has four distinct roots: $-1/3$, $1/4$, $1/2$ and 2.

Answer: 4. [The roots are $-1/3$, $1/4$, $1/2$ and 2.]