BC Exam

Texas A&M Math Contest

2 November, 2024

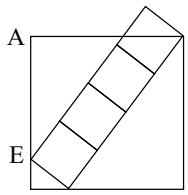
(NOTE: If units are appropriate, please include them in your answer. All answers must be simplified where possible.)

1. All positive integers are written consecutively (starting from 1) as a single sequence of decimal digits. Find the 2024th digit in that sequence.

2. In triangle $\triangle ABC$, $AB = 3$, $AC = 5$, and the angle $\angle ABC$ is double the angle $\angle ACB$. Find the length of side $\overline{BC}$.

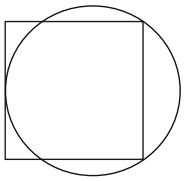
3. In a soccer tournament, every two teams played each other twice. What was the number of participating teams if the total number of games played was 182?

4. The figure below shows a configuration of one large square and four smaller squares. Find the edge length of a smaller square if $AE = 13$.

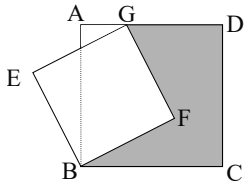


5. Find the minimal possible value of the expression $x + \dfrac{2}{x}$, where $x > 0$.

6. A circle is tangent to one side of a square and passes through two other vertices of the square, as shown in the figure below. If the square's side length is 8, find the area of the circle.

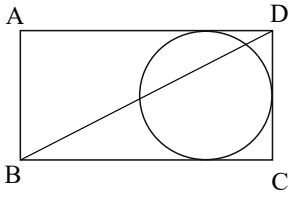


7. Two squares, $ABCD$ and $EBFG$, are positioned as shown in the figure below. If $\overline{AB}$ bisects $\overline{EG}$ and $EG = 2$, find the area of the shaded region.



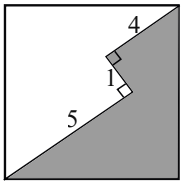
8. Find a positive number x such that $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x}}}$.

9. A circle is tangent to three sides of a rectangle $ABCD$, as shown in the figure. A diagonal intersects the circle at two points, forming a chord. If $BC = 4\sqrt{3}$ and $CD = 4$, find x^2, where x is the length of the chord.



10. Consider a triangle with sides of lengths 7, 8, and 9. A circle with radius 1 rolls along the triangle's interior, always tangent to at least one side as it rolls along. Determine the length of the path traced by the center of the circle as it completes one full revolution around the triangle's interior. Reduce and rationalize the denominator of your final answer.

11. Inside a square, a pair of opposite sides is connected by three line segments with lengths 5, 1, and 4, in that order, as shown below. Find the area of the shaded region.



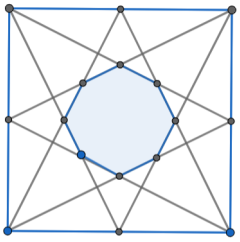
12. Let c be a real solution of the equation $x^4 - 3x + 1 = 0$. Evaluate the expression $c^6 + c^4 - 3c^3 + c^2 - 3c$.

13. In how many ways can you draw four diagonals inside a convex heptagon, without intersecting each other, to divide it into five triangles, such that each triangle shares at least one side with the heptagon?

14. Consider a fraction $\dfrac{6n-1}{7n+1}$, where n is a positive integer. Find the smallest value of n for which the fraction is not in lowest terms.

15. All real solutions of the inequality $\sqrt{3-2x-x^2} > x+1$ fill an interval of the real line. Find the length of that interval.

16. From each vertex of a square with side length 1, draw a line segment to the midpoint of the opposite side's adjacent edges, as shown in the figure. Find the area of the shaded octagon formed by these 8 line segments.



17. Consider a rectangular box with side lengths 1, 2, and 3. A plane cuts through the box, passing through the two opposite vertices A and G and containing a shortest path between these vertices on the box's surface. Find the area of the cross-section formed by this plane.

18. How many distinct real roots does the following equation have?

$$(2x^2-5x+2)^3+(6x^2-x-1)^3=(8x^2-6x+1)^3$$

19. Consider points $P_1, P_2, \ldots, P_{100}$ on side $\overline{BC}$ of an isosceles triangle $\triangle ABC$ with $AB = AC = 2$. For each point P_i, define $k_i = AP_i^2 + BP_i \times P_iC$ for $i = 1, 2, \ldots, 100$. Find the sum $k_1 + k_2 + \cdots + k_{100}$.

20. Inside rectangle $ABCD$ with $AB = 1$, a semicircle O_1 is tangent to side $\overline{AD}$ and to two other circles, O_2 and O_3, as shown in the figure. Circle O_2 is tangent to sides $\overline{AB}$ and $\overline{BC}$, as well as to circle O_3. Circle O_3 is tangent to sides $\overline{BC}$ and $\overline{AC}$. Given that $\overline{AP}$ is the diameter of semicircle O_1, find the radius of O_2.