Best Student Exam
Texas A&M High School Math Contest
November 2, 2024

1. A point in the plane, both of whose coordinates are integers with absolute value less than or equal to 4, is chosen at random, with all such points having an equal probability of being chosen. What is the probability that the distance from the point to the origin is at most 3?

2. For which values of $k \neq 0$ is the line $y = kx - 2k$ tangent to the circle $x^2 + y^2 = 2k^2$?

3. Find all solutions (x, y, z) of the following system of equations:

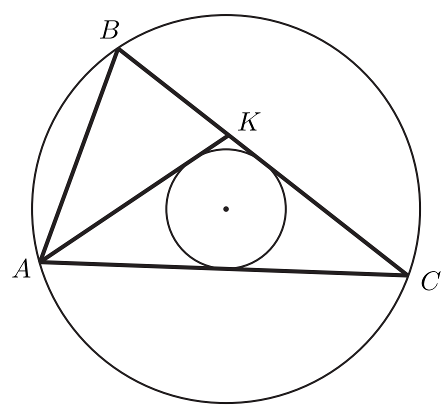
$$\begin{cases} x + y + xy &= 19 \\ y + z + yz &= 11 \\ z + x + zx &= 14 \end{cases}$$

4. Sides a, b, c of a triangle satisfy the equality

$$\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}.$$

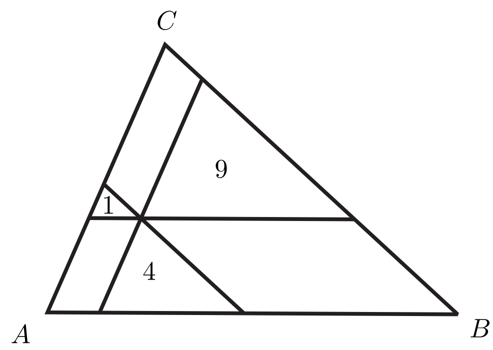
Find an angle of the triangle (in degrees).

5. Let AK be an angle bisector in triangle $\triangle ABC$ (where K belongs to the side BC). The center of the circle inscribed in $\triangle AKC$ coincides with the center of the circle circumscribed around $\triangle ABC$. Find $\angle ACB$ in degrees.



6. Find the minimal value of $|\cos x| + |\cos 2x|$.

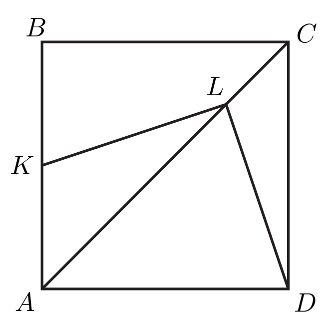
7. Three lines parallel to the sides of triangle $\triangle ABC$ are drawn through a point inside the triangle. They form together with the sides of $\triangle ABC$ three parallelograms and three triangles. Suppose that the areas of the triangles are 1, 4, and 9. What is the area of $\triangle ABC$?



8. How many pairs (x, y) of integers satisfy the equation

$$x^2(y-1) + y^2(x-1) = 1$$

9. In a square $ABCD$, let K be the midpoint of the side AB and let L be a point on the diagonal AC such that the angle $\angle KLD$ is a right angle. Find $\frac{AL}{LC}$.



10. Evaluate the integral $\int_0^{\pi} \sqrt{1+\cos 2x}\; dx$.

11. Evaluate the integral $\int_0^{\pi/2} \left(\cos^2(\cos x) + \sin^2(\sin x)\right)\; dx$.

12. Let $ABCD$ be a trapezoid, where $AD \parallel BC$. Suppose that the diagonals AC and BD are perpendicular, the height of $ABCD$ is equal to 4, and one of the diagonals has length 5. What is the area of the trapezoid?

13. How many triples of integers satisfy the inequality $a^2 + b^2 + c^2 < ab + 3b + 2c$.

14. What is the fifth decimal place of $(1.0025)^{10}$?

15. Find all pairs of positive integers x and y such that $x(x+1)(x+7)(x+8) = y^2$.

16. Find the maximal value of the product $a_1a_2\cdots a_n$, where positive integers n and $a_1, a_2, \ldots, a_n$ are such that $a_1 + a_2 + \cdots + a_n = 2024$.

17. Given that a, b, c, d, e are real numbers such that

$$\begin{cases} a+b+c+d+e & = & 8, \\ a^2+b^2+c^2+d^2+e^2 & = & 16, \end{cases}$$

determine the maximal value of e.

18. Find all pairs of primes (p, q) such that $p+q=(p-q)^3$.

19. It is known that the number S satisfies the following condition: if $a+b+c+d=S$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=S$ for some numbers a, b, c, d different from 0 and 1, then $\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1}=S$. Find S.

20. Let T_n be the sequence given by $T_1 = 2$ and $T_{n+1} = T_n^2 - T_n + 1$. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{T_n}$.