Best Student Exam
Texas A&M High School Math Contest
November 2, 2024

1. A point in the plane, both of whose coordinates are integers with absolute value less than or equal to 4, is chosen at random, with all such points having an equal probability of being chosen. What is the probability that the distance from the point to the origin is at most 3?

Solution: There are 9 integers in the interval $[-4, 4]$. Therefore, the total number of points whose coordinates are integers with absolute value less than or equal to 4 is 81. A point (x, y) is on distance at most 3 from the origin if and only if $x^2 + y^2 \leq 9$. We get the following possible (unordered) pairs of absolute values of coordinates $(3, 0)$, $(2, 0)$, $(1, 0)$, $(0, 0)$, $(2, 1)$, $(1, 1)$, $(2, 2)$. The number of points with these pairs of absolute values of coordinates are 4, 4, 4, 1, 8, 4, 4, respectively. Therefore, there are 29 points a distance at most 3 from the origin. It follows that the probability is 29/81.

Answer: $\frac{29}{81}$

2. For which values of $k \neq 0$ is the line $y = kx - 2k$ tangent to the circle $x^2 + y^2 = 2k^2$?

Solution: We are asked to find all real k for which the system

$$\begin{cases} x^2 + y^2 & = & 2k^2 \\ y & = & kx - 2k \end{cases}$$

has a unique solution.

Substituting $y = kx - 2k$ into the first equation, we get $x^2 + (kx - 2k)^2 = 2k^2$, i.e., $(k^2 + 1)x^2 - 4k^2x + 2k^2 = 0$. This equation (and thus the system) has unique solution if and only if its discriminant is equal to zero. The discriminant is equal to $16k^4 - 8k^2(k^2 + 1) = 8k^4 - 8k^2 = 8k^2(k^2 - 1)$.

Answer: $k = 1, -1$.

3. Find all solutions (x, y, z) of the following system of equations:

$$\begin{cases} x + y + xy & = & 19 \\ y + z + yz & = & 11 \\ z + x + zx & = & 14 \end{cases}$$

Solution: Adding 1 to both sides of each equation, and factoring the left-hand side, we get

$$\begin{cases} (x + 1)(y + 1) & = & 20 \\ (y + 1)(z + 1) & = & 12 \\ (z + 1)(x + 1) & = & 15 \end{cases}$$

Multiplying all equations together, we get $(x + 1)^2(y + 1)^2(z + 1)^2 = 3600$, hence $(x + 1)(y + 1)(z + 1) = \pm 60$. Dividing this equation by each of the equations of the last system, we get

$$\begin{cases} z + 1 & = & \pm 3 \\ x + 1 & = & \pm 5 \\ y + 1 & = & \pm 4 \end{cases}$$

where the right-hand sides are either all positive or all negative. We get hence two solutions $(4,3,2)$ and $(-6,-5,-4)$.

Answer: $(4,3,2)$ and $(-6,-5,-4)$.

4. Sides a, b, c of a triangle satisfy the equality

$$\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}.$$

Find an angle of the triangle (in degrees).

Solution: Adding the fractions, we get

$$\frac{a+2b+c}{ab+ac+b^2+bc} = \frac{3}{a+b+c}.$$

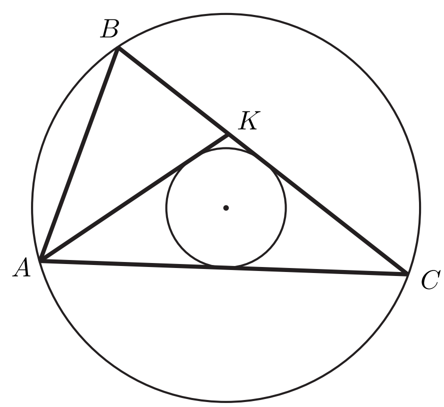
After simplification, we get

$$b^2 = a^2 - ac + c^2,$$

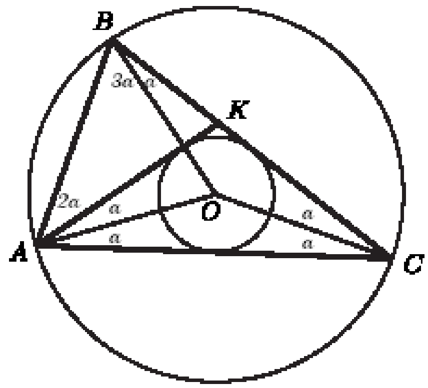
which by the Law of Cosines implies that the angle opposite to the side of length b is equal to $60°$.

Answer: $60°$

5. Let AK be an angle bisector in triangle $\triangle ABC$ (where K belongs to the side BC). The center of the circle inscribed in $\triangle AKC$ coincides with the center of the circle circumscribed around $\triangle ABC$. Find $\angle ACB$ in degrees.



Solution: Let O be the common center of the circle inscribed in $\triangle AKC$ and the circle circumscribed around $\triangle ABC$. Then O is the intersection point of the bisectors of $\triangle AKC$, and the segments AO, BO, and CO are equal. Denote $\angle KAO = \alpha$. Then $\angle OAC = \alpha$. Since $AO = OC$, we have $\angle OCA = \angle OAC = \alpha$. Since CO is the bisector of $\angle ACB$, we have $\angle OCB = \angle OCA = \alpha$. Since $OB = OC$, $\angle OBC = \angle OCB = \alpha$. Since AK is the bisector of $\angle BAC$, we have $\angle BAK = \angle KAC = 2\alpha$, hence $\angle BAO = 3\alpha$. Since $AO = OB$, we have $\angle ABO = \angle BAO = 3\alpha$.



We conclude that the angles of $\triangle ABC$ are $\angle A = 4\alpha$, $\angle B = 4\alpha$, $\angle C = 2\alpha$. Their sum 10α is equal to $180°$, hence $\alpha = 18°$. Consequently, $\angle ACB = 36°$.

Answer: $36°$

6. Find the minimal value of $|\cos x| + |\cos 2x|$.

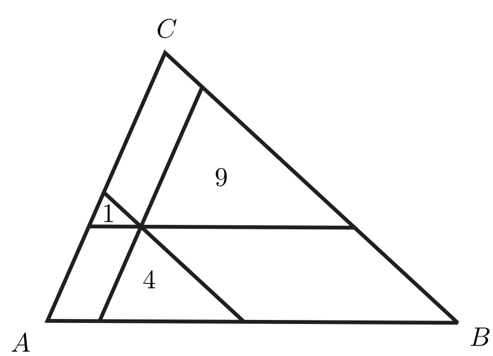
Solution: Denote $|\cos x| = t$. Then $|\cos x| + |\cos 2x| = t + |2t^2 - 1| = f(t)$, where $0 \le t \le 1$. We have $f(t) = -2t^2 + t + 1$ for $0 \le t < \sqrt{2}/2$, and $f(t) = 2t^2 + t - 1$ for $\sqrt{2}/2 \le t \le 1$.

The polynomial $f(t) = -2t^2 + t + 1$ is increasing on $(-\infty, 1/4]$ and decreasing on $[1/4, \infty)$. Since $0 < 1/4 < \sqrt{2}/2$, the minimal value of the polynomial on the interval $[0, \sqrt{2}/2]$ is $\min\{f(0), f(\sqrt{2}/2)\} = \min\{1, \sqrt{2}/2\} = \sqrt{2}/2$.

The polynomial $2t^2 + t - 1$ is decreasing on $(-\infty, -1/4]$ and increasing on $[-1/4, \infty)$. Consequently, it is increasing on the interval $[\sqrt{2}/2, 1]$, so its minimal value on this interval is its value in $\sqrt{2}/2$, which is $\sqrt{2}/2$.

Answer: $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$.

7. Three lines parallel to the sides of triangle $\triangle ABC$ are drawn through a point inside the triangle. They form together with the sides of $\triangle ABC$ three parallelograms and three triangles. Suppose that the areas of the triangles are 1, 4, and 9. What is the area of $\triangle ABC$?



Solution: Let a, b, c be the lengths of the sides of the triangle, and let S_a, S_b, S_c be the areas of the three triangles adjacent to them, respectively. We will find a general formula for the area of $\triangle ABC$ for all triples of areas of the small triangles. Let h_a, h_b, h_c be the heights of the small triangles from their common vertex to the sides contained in the sides of lengths a, b, c, respectively. Let H_a, H_b, H_c be the heights of $\triangle ABC$ to the sides of the corresponding lengths. Let S be the area of $\triangle ABC$. Then, it follows from similarity of triangles that $h_a^2 : H_a^2 = S_a : S$, $h_b^2 : H_b^2 = S_b : S$, and $h_c^2 : H_c^2 = S_c : S$. We also have $S = aH_a/2 = bH_b/2 = cH_c/2$, and $S = (ah_a + bh_b + ch_c)/2$.

Replacing h_a, h_b, h_c by $\frac{H_a\sqrt{S_a}}{\sqrt{S}}$, $\frac{H_b\sqrt{S_b}}{\sqrt{S}}$, $\frac{H_c\sqrt{S_c}}{\sqrt{S}}$, respectively, we get

$$S = \frac{1}{2\sqrt{S}}\left(aH_a\sqrt{S_a} + bH_b\sqrt{S_b} + cH_c\sqrt{S_c}\right) = \frac{1}{\sqrt{S}}\left(S\sqrt{S_a} + S\sqrt{S_b} + S\sqrt{S_c}\right),$$

hence $\sqrt{S} = \sqrt{S_a} + \sqrt{S_b} + \sqrt{S_c}$, i.e.,

$$S = \left(\sqrt{S_a} + \sqrt{S_b} + \sqrt{S_c}\right)^2.$$

In our case, we get $S = (1+2+3)^2 = 36$.

Answer: 36.

8. How many pairs (x, y) of integers satisfy the equation

$$x^2(y-1) + y^2(x-1) = 1.$$

Solution: We can rewrite the equation as $(x+y+2)(xy-x-y+2) = 5$. It follows that an integer solution of the equation must be a solution of one of the systems:

$$\begin{cases} x+y+2 &= 1 \\ xy-x-y+2 &= 5 \end{cases} \quad \begin{cases} x+y+2 &= -1 \\ xy-x-y+2 &= -5 \end{cases}$$
$$\begin{cases} x+y+2 &= 5 \\ xy-x-y+2 &= 1 \end{cases} \quad \begin{cases} x+y+2 &= -5 \\ xy-x-y+2 &= -1 \end{cases}$$

which are equivalent to

$$\begin{cases} x+y &= -1 \\ xy &= 2 \end{cases} \quad \begin{cases} x+y &= -3 \\ xy &= -10 \end{cases}$$
$$\begin{cases} x+y &= 3 \\ xy &= 2 \end{cases} \quad \begin{cases} x+y &= -7 \\ xy &= -10 \end{cases}$$

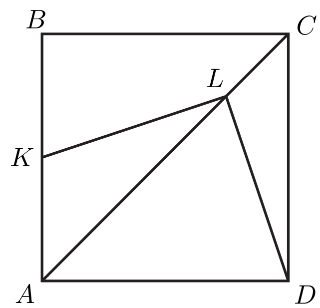
It follows that x and y are two roots of one of the following quadratic equations

$$t^2 + t + 2 = 0, \quad t^2 + 3t - 10 = 0, \quad t^2 - 3t + 2 = 0, \quad t^2 + 7t - 10 = 0.$$

Their discriminants are $-7, 49, 1, 89$, respectively. It follows that x and y are roots either of the second or of the third equations, hence $\{x, y\} = \{-5, 2\}$ or $\{x, y\} = \{1, 2\}$.

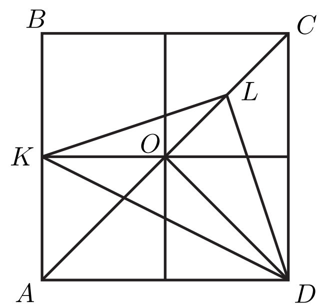
Answer: 4 pairs

9. In a square $ABCD$, let K be the midpoint of the side AB and let L be a point on the diagonal AC such that the angle $\angle KLD$ is a right angle. Find $\frac{AL}{LC}$.



Solution: Note that a point L such that $\angle KLD = 90°$ is an intersection of the circle with diameter KD and the line AC. One of these intersections is A, so the point L is unique. Let us prove that if $\frac{AL}{LC} = 3$, then $\angle KLD = 90°$. By uniqueness of L, this will show that if $\angle KLD = 90°$, then $\frac{AL}{LC} = 3$.

Let us divide the square $ABCD$ into four congruent squares as on the figure. Then the point L on AC such that $\frac{AL}{LC} = 3$ is the center of the square adjacent to C.



It is easy to see by symmetry that $KL = BL$ and $BL = LD$. Consequently $\triangle KLD$ is isosceles. Let a be the length of the side of the square $ABCD$. Then $OD = OC = \frac{a\sqrt{2}}{2}$, hence $OL = \frac{a\sqrt{2}}{4}$. By Pythagorean theorem for $\triangle LOD$, we get $LD = KL = \sqrt{\frac{a^2}{8} + \frac{a^2}{2}} = \frac{a\sqrt{5}}{2\sqrt{2}}$. By Pythagorean theorem for $\triangle KAD$, we get $KD = \sqrt{\frac{a^2}{4} + a^2} = \frac{a\sqrt{5}}{2}$.

We have $LD\sqrt{2} = KL\sqrt{2} = KD$, hence $\angle KLD = 90°$.

Answer: 3.

10. Evaluate the integral $\int_0^\pi \sqrt{1+\cos 2x}\, dx$.

Solution: We have $\sqrt{1+\cos 2x} = \sqrt{2\cos^2 x} = \sqrt{2} \cdot |\cos x|$. Therefore,

$$\int_0^\pi \sqrt{1+\cos 2x}\, dx = \sqrt{2}\left(\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right) = 2\sqrt{2}.$$

Answer: $2\sqrt{2}$

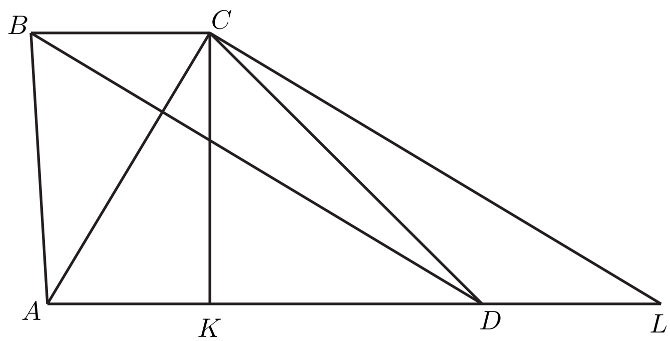
11. Evaluate the integral $\int_0^{\pi/2} \left(\cos^2(\cos x) + \sin^2(\sin x)\right)\, dx$.

Solution: We can transform $\int_0^{\pi/2} \sin^2(\sin x)\, dx$ by substitution $y = \frac{\pi}{2} - x$ to $-\int_{\pi/2}^0 \sin^2(\cos y)\, dy = \int_0^{\pi/2} \sin^2(\cos x)\, dx$. Consequently, the original integral is equal to $\int_0^{\pi/2} \cos^2(\cos x) + \sin^2(\cos x)\, dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$.

Answer: $\frac{\pi}{2}$

12. Let $ABCD$ be a trapezoid, where $AD \parallel BC$. Suppose that the diagonals AC and BD are perpendicular, the height of $ABCD$ is equal to 4, and one of the diagonals has length 5. What is the area of the trapezoid?

Solution: Without loss of generality, let $AC = 5$. Let CK be the height of the trapezoid. Then $CK = 4$, so $AK = 3$. Let CL be the line parallel to BD, where L is on the line AD.



Then AC is perpendicular to CL, and $\triangle ACK$ is similar to $\triangle CLK$. Consequently, $\frac{KL}{CK} = \frac{CK}{AK}$, i.e., $\frac{KL}{4} = \frac{4}{3}$, hence $KL = \frac{16}{3}$.

Since opposite sides BC and DL of the parallelogram $BCLD$ are equal, the area of the trapezoid is equal to the area of $\triangle ACL$, which is equal to $\frac{1}{2} \cdot CK \cdot AL = \frac{1}{2} \cdot 4 \cdot \left(3 + \frac{16}{3}\right) = \frac{50}{3}$.

Answer: $\frac{50}{3} = 16\frac{2}{3}$

13. How many triples of integers satisfy the inequality $a^2 + b^2 + c^2 < ab + 3b + 2c$.

Solution: We can write the inequality as $(a - b/2)^2 + \frac{3}{4}(b-2)^2 + (c-1)^2 < 4$, which is equivalent to $(2a-b)^2 + 3(b-2)^2 + 4(c-1)^2 < 16$.

The solution of a system

$$\begin{cases} 2a - b &= t_1 \\ b - 2 &= t_2 \\ c - 1 &= t_3 \end{cases}$$

where t_1, t_2, t_3 are parameters, is

$$\begin{cases} a &= (t_1 + t_2)/2 + 1 \\ b &= t_2 + 2 \\ c &= t_3 + 1 \end{cases}$$

Consequently, the numbers a, b, c are integers if and only if t_1, t_2, t_3 are integers and t_1, t_2 have the same parity. For each such triple (t_1, t_2, t_3) we will get a unique solution (a, b, c).

Consequently, the number of integer solutions of the original inequality is equal to the number of integer solutions of the inequality $t_1^2 + 3t_2^2 + 4t_3^2 < 16$ such that t_1 and t_2 have the same parity. We have $t_1^2 \in \{0, 1, 4, 9\}$, $3t_2^2 \in \{0, 3, 12\}$, and $4t_3^2 \in \{0, 4\}$. Taking into account that t_1, t_2 are of the same parity, we get the following triples $(t_1^2, 3t_2^2, 4t_3^2)$:

$$(0,0,0), (0,0,4), (0,12,0), (1,3,0), (1,3,4), (4,0,0), (4,0,4), (9,3,0).$$

The corresponding values of (t_1, t_2, t_3) are

$$(0,0,0),(0,0,\pm 1),(0,\pm 2,0),(\pm 1,\pm 1,0),(\pm 1,\pm 1,\pm 1),(\pm 2,0,0),(\pm 2,0,\pm 1),(\pm 3,\pm 1,0),$$

where signs are independent. It follows that the number of triples (a,b,c) satisfying the original inequality is

$$1+2+2+4+8+2+4+4=27.$$

Answer: 27

14. What is the fifth decimal place of $(1.0025)^{10}$.

Solution: We have $(1.0025)^{10} = 1 + \frac{10}{400} + \frac{45}{400^2} + \frac{120}{400^3} + \frac{210}{400^4} + \frac{252}{400^5} + \frac{210}{400^6} + \frac{120}{400^7} + \frac{45}{400^8} + \frac{10}{400^9} + \frac{1}{400^{10}}$

We have $\frac{210}{400^4} < \frac{1}{400^3}$, $\frac{252}{400^5} < \frac{1}{400^4}$, $\frac{210}{400^6} < \frac{1}{400^5}$, $\frac{120}{400^7} < \frac{1}{400^6}$, $\frac{45}{400^8} < \frac{1}{400^7}$, $\frac{10}{400^9} < \frac{1}{400^8}$, so

$$\frac{210}{400^4}+\frac{252}{400^5}+\frac{210}{400^6}+\frac{120}{400^7}+\frac{45}{400^8}+\frac{10}{400^9}+\frac{1}{400^{10}} < \frac{1}{400^3}+\frac{1}{400^4}+\frac{1}{400^5}+\frac{1}{400^6}+\frac{1}{400^7}+\frac{1}{400^8}+\frac{1}{400^{10}}$$

The right-hand side is equal to

$$\frac{1-\frac{1}{400^6}}{400^2\cdot 399} + \frac{1}{400^{10}} < 10^{-7}.$$

The first four summands are

$$1+\frac{250}{10^4}+\frac{45\cdot 625}{10^8}+\frac{120\cdot 15,625}{10^{12}} = 1+0.025+0.00028125+0.000001875 = 1.025283125.$$

Since the rest of the sum is less than 10^{-7}, it will not change the fifth digit, so it is equal to 8.

Answer: 8

15. Find all pairs of positive integers x and y such that $x(x+1)(x+7)(x+8) = y^2$.

Solution: The right-hand side is equal to $x^4+16x^3+71x^2+56x$. It is greater than $(x^2+8x)^2 = x^4+16x^3+64x^2$ and less than $(x^2+8x+4)^2 = x^4+16x^3+72x^2+64x+16$. It follows that y is equal to one of the numbers $x^2+8x+1, x^2+8x+2, x^2+8x+3$. In the first case, we have

$$x^4+16x^3+71x^2+56x = x^4+16x^3+66x^2+16x+1,$$

hence

$$5x^2+40x-1=0,$$

which has no integer roots. In the second case, we have

$$x^4+16x^3+71x^2+56x = x^4+16x^3+68x^2+32x+4,$$

hence $3x^2 + 24x - 4 = 0$, which also has no integer roots. In the third case, we have

$$x^4 + 16x^3 + 71x^2 + 56x = x^4 + 16x^3 + 70x^2 + 48x + 9,$$

i.e., $x^2+8x-9 = 0$, which has roots 1 and -9. If $x = 1$, then $x(x+1)(x+7)(x+8) = 144 = 12^2$. So, the only positive integer solution is $(x, y) = (1, 12)$.

Answer: $(1, 12)$

16. Find the maximal value of the product $a_1a_2 \cdots a_n$, where positive integers n and $a_1, a_2, \ldots, a_n$ are such that $a_1 + a_2 + \cdots + a_n = 2024$.

Solution: Suppose that $a_1, a_2, \ldots, a_n$ and n realize the maximal value of $a_1a_2 \cdots a_n$.

If a_i is even and greater than 4, then we can replace it by $a_i/2, a_i/2$ and, having the same sum, get a greater product, since $(a_i/2)^2 > a_i$ for $a_i > 4$.

If a_i is odd and greater than 3, then we can replace a_i by $(a_i - 1)/2, (a_i + 1)/2$, and replace a_i in the product by $(a_i^2 - 1)/4 > a_i$.

If one of a_i is equal to 1, then we can replace two of numbers a_i, a_j by $a_j + 1$ and get a larger product.

Consequently, all numbers a_i belong to the set $\{2, 3, 4\}$.

Note that $2 \times 2 \times 2 < 3 \times 3$, so we can not have more than two instances of 2.

We have $2 \times 4 < 3 \times 3$, so we can not have 2 and 4 simultaneously.

We have $4 \times 4 < 3 \times 3 \times 2$, so we can not have more than one 4.

Consequently, all numbers a_i are equal to 3, except for maybe one 4, or one 2, or two 2. Since 2024 gives remainder 2 when divided by 3, in our case we have all a_i equal to 3 except for one equal to 2. Consequently, the maximal product is $2 \cdot 3^{674}$.

Answer: $2 \cdot 3^{674}$

17. Given that a, b, c, d, e are real numbers such that

$$\begin{cases} a + b + c + d + e & = & 8, \\ a^2 + b^2 + c^2 + d^2 + e^2 & = & 16, \end{cases}$$

determine the maximal value of e.

Solution: Using the inequality between the arithmetic and quadratic mean, we get $e^2 = 16 - (a^2 + b^2 + c^2 + d^2) \leq 16 - \frac{(a+b+c+d)^2}{4} = 16 - \frac{(8-e)^2}{4}$.

Consequently, $5e^2 - 16e \leq 0$, hence $e \in [0, 16/5]$.

The conditions of the problem are satisfied for $a = b = c = d = \frac{6}{5}$ and $e = \frac{16}{5}$, hence the value $\frac{16}{5}$ is achieved, so it is the maximal value of e.

Answer: $\frac{16}{5} = 3.2$.

18. Find all pairs of primes (p, q) such that $p + q = (p - q)^3$.

Solution: We have $(p - q)^3 \equiv p - q \pmod 3$. Consequently, $p + q \equiv p - q \pmod 3$, hence q is divisible by 3, i.e., $q = 3$. We get $p + 3 = (p - 3)^3$, i.e.,

$$p^3 - 9p^2 + 26p - 30 = 0.$$

One of the roots is $p = 5$. Dividing the polynomial on the left-hand side by $p - 5$, we get $p^2 - 4p + 6 = 0$, which does not have real roots. Consequently, the only solution is $(p, q) = (5, 3)$.

Answer: $(5, 3)$

19. It is known that the number S satisfies the following condition: if $a+b+c+d = S$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} = S$ for some numbers a, b, c, d different from 0 and 1, then $\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1} = S$. Find S.

Solution: If the conditions $a + b + c + d = S$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = S$ are satisfied for (a, b, c, d), then they are satisfied for $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right)$. So, the conclusion must be also satisfied for them, i.e., we must have

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} + \frac{d}{1-d} = S.$$

Adding this to $\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} = S$, we get $-4 = 2S$, hence $S = -2$. Note that it is not required to check if the statement is true for $S = -2$ (but it is).

Answer: $S = -2$.

20. Let T_n be the sequence given by $T_1 = 2$ and $T_{n+1} = T_n^2 - T_n + 1$. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{T_n}$.

Solution: Let us prove by induction that $\sum_{k=1}^{n} \frac{1}{T_k} = \frac{T_{n+1}-2}{T_{n+1}-1}$.
We have $T_2 = 4 - 2 + 1 = 3$ and $\frac{1}{T_1} = \frac{1}{2} = \frac{3-2}{3-1}$.
If the statement is true for n, then

$$\sum_{k=1}^{n+1} \frac{1}{T_k} = \frac{T_{n+1} - 2}{T_{n+1} - 1} + \frac{1}{T_{n+1}} = \frac{T_{n+1}^2 - 2T_{n+1} + T_{n+1} - 1}{T_{n+1}^2 - T_{n+1}} = \frac{T_{n+1}^2 - T_{n+1} - 1}{T_{n+1}^2 - T_{n+1}} = \frac{T_{n+2} - 2}{T_{n+2} - 1}.$$

Since we have $x^2 - x + 1 > 2x$ for $x > 3$, the sequence T_n converges to infinity. Consequently, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{T_n} = 1$.

Answer: 1