CD Exam

Texas A&M Math Contest

2 November, 2024

(NOTE: If units are appropriate, please include them in your answer. All answers must be simplified where possible.)

1. Evaluate $1 - 2 + 3 - 4 + 5 - 6 + \cdots + 2023 - 2024$.

2. The function f has the property that, for each real number x in its domain,

$$f(x) + f\left(\frac{1}{x}\right) = x$$

   What is the largest possible domain of f?

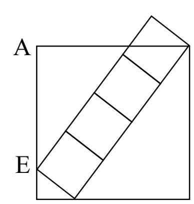
3. Find a real number b such that the equation $|x + 20| + |x + 3| = b$ has infinitely many solutions.

4. What is the smallest positive integer n such that n is divisible by 24, n^2 is a perfect cube, and n^3 is a perfect square? You may leave your answer in the form of an exponent.

5. Suppose that $(3^a)^b = 3^a \times 3^b$. If $b = 5$, find a.

6. In triangle $\triangle ABC$, $AB = 3$, $AC = 5$, and the angle $\angle ABC$ is double the angle $\angle ACB$. Find the length of side $\overline{BC}$.

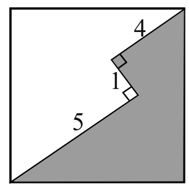
7. The figure below shows a configuration of one large square and four smaller squares. Find the edge length of a smaller square if $AE = 13$.



8. For any real number x, we have $3f(x) + f(8 - x) = x^3$. Find $f(4)$

9. Consider a triangle with sides of lengths 7, 8, and 9. A circle with radius 1 rolls along the triangle's interior, always tangent to at least one side as it rolls along. Determine the length of the path traced by the center of the circle as it completes one full revolution around the triangle's interior. Reduce and rationalize the denominator of your final answer.

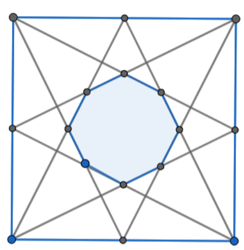
10. Inside a square, a pair of opposite sides is connected by three line segments with lengths 5, 1, and 4, in that order, as shown below. Find the area of the shaded region.



11. Let p be the greatest prime factor of 9991. Compute the sum of the digits of p.

12. In how many ways can you draw four diagonals inside a convex heptagon, without intersecting each other, to divide it into five triangles, such that each triangle shares at least one side with the heptagon?

13. From each vertex of a square with side length 1, draw a line segment to the midpoint of the opposite side's adjacent edges, as shown in the figure. Find the area of the shaded octagon formed by these 8 line segments.



14. Let A, B, and C be three distinct points on the graph of $y = x^2$, with $\overline{AB}$ parallel to the x-axis and $\triangle ABC$ a right triangle with area 2024. What is the sum of the digits of the y-coordinate of C?

15. Let a, b, c and d be real numbers. Let $P(x) = ax^9 + bx^5 + cx^3 + dx + 8$. Suppose $P(-3) = 9$. Find $P(3)$.

16. Given $1001 = abc + ab + ac + bc + a + b + c + 1$, where a, b, c are positive integers. Compute abc.

17. Consider a rectangular box with side lengths 1, 2, and 3. A plane cuts through the box, passing through the two opposite vertices A and G and containing a shortest path between these vertices on the box's surface. Find the area of the cross-section formed by this plane.

18. Let c, d, and e be real numbers such that the polynomial $x^4 - 26x^3 + cx^2 + dx + e$ has four roots which are consecutive integers. What is e?

19. Consider points $P_1, P_2, \ldots, P_{100}$ on side BC of an isosceles triangle $\triangle ABC$ with $AB = AC = 2$. For each point P_i, define $k_i = AP_i^2 + BP_i \times P_iC$ for $i = 1, 2, \ldots, 100$. Find the sum $k_1 + k_2 + \cdots + k_{100}$

20. Inside rectangle $ABCD$ with $AB = 1$, a semicircle O_1 is tangent to side $\overline{AD}$ and to two other circles, O_2 and O_3, as shown in the figure. Circle O_2 is tangent to sides $\overline{AB}$ and $\overline{BC}$, as well as to circle O_3. Circle O_3 is tangent to sides $\overline{BC}$ and $\overline{AC}$. Given that $\overline{AP}$ is the diameter of semicircle O_1, find the radius of O_2.