Solutions to CD Exam

Texas A&M High School Math Contest

2 November, 2024

1. Evaluate $1 - 2 + 3 - 4 + 5 - 6 + \cdots + 2023 - 2024$.

   Solution.

   $1-2+3-4+5-6+...+2023-2024 = (1-2)+(3-4)+(5-6)+...+(2023-2024) = (-1)(1012) = -1012$

   Answer: -1012

2. The function f has the property that, for each real number x in its domain,

   $$f(x) + f\left(\frac{1}{x}\right) = x$$

   What is the largest possible domain of f?

   Solution. Let a be any real number in the domain of f. Then we have

   $$f(a) + f\left(\frac{1}{a}\right) = f\left(\frac{1}{a}\right) + f(a)$$

   $$a = \frac{1}{a}$$

   So the only possible values of a are ± 1.

   Answer: $\{-1, 1\}$

3. Find a real number b such that the equation $|x + 20| + |x + 3| = b$ has infinitely many solutions.

   Solution. $|x + 20| + |x + 3|$ equals the sum of the distances from the point x to -20 and -3 on the real number line. For $-20 \leq x \leq -3$, the sum is 17. Therefore $b = 17$.

   Answer: 17

4. What is the smallest positive integer n such that n is divisible by 24, n^2 is a perfect cube, and n^3 is a perfect square? You may leave your answer in the form of an exponent.

   Solution. We know that $n^2 = k^3$ for some positive integer k, and $n^3 = m^2$ for some positive integer m. This means that

   $$n^6 = k^9 \text{ and } n^6 = m^4$$

   To satisfy both equations, $n^6 = a^{36}$ for some positive integer a, so $n = a^6$. Since n is divisible by 24, a is divisible by 2 and 3, the prime factors of 24, so the smallest possible value of a is 6. $n = 6^6$.

   Answer: 6^6 or 36^3 or 216^2 or 46656

5. Suppose that $(3^a)^b = 3^a \times 3^b$. If $b = 5$, find a.

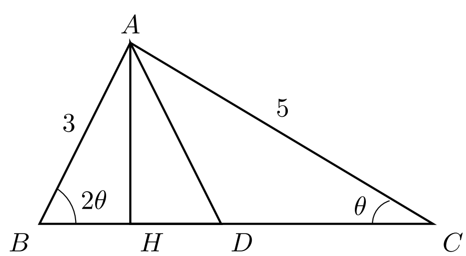
Solution. We have

$$\begin{aligned} (3^a)^b &= 3^a \times 3^b \\ ab &= a + b \\ 5a &= a + 5 \\ a &= \frac{5}{4} \end{aligned}$$

Answer: $\frac{5}{4}$

6. In triangle $\triangle ABC$, $AB = 3$, $AC = 5$, and the angle $\angle ABC$ is double the angle $\angle ACB$. Find the length of side $\overline{BC}$.

Solution. As shown below, draw a line segment $\overline{BD}$ to have $AD = 3$. Since $\angle ADB = 2\theta$, $\angle DAC = \theta$, and hence $\triangle DAC$ is isosceles. This implies $AD = DC = 3$.



Let H be the foot of the perpendicular from A to $\overline{BC}$, and let $BH = x$. Applying the Pythagorean theorem in $\triangle ABH$ and $\triangle ACH$, we have:

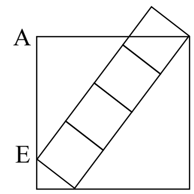
$$3^2 - x^2 = 5^2 - (3 + x)^2 \quad \text{or} \quad x = \frac{7}{6}.$$

Thus,

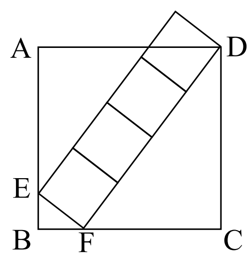
$$BC = 2x + 3 = 2 \times \frac{7}{6} + 3 = \frac{16}{3}.$$

Answer: $\frac{16}{3}$

7. The figure below shows a configuration of one large square and four smaller squares. Find the edge length of a smaller square if $AE = 13$.



Solution. We use the similarity of triangles in the following figure. Let $EB = x$. Then $AB = CD = 13 + x$.



Since $\angle EFB = \angle FDC$, the two triangles $\triangle EFB$ and $\triangle FDC$ are similar with the ratio:

$$EF : FD = 1 : 4.$$

This implies $FC = EB \times 4 = 4x$, and so

$$BF : CD = 13 + x - 4x : 13 + x = 1 : 4 \quad \Rightarrow \quad x = 3.$$

Thus, $BF = 4$ and $EF = 5$.

Answer: 5

8. For any real number x, we have $3f(x) + f(8 - x) = x^3$. Find $f(4)$

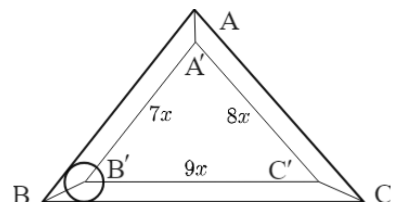
Solution. We have

$$3f(4) + f(4) = 4^3$$
$$4f(4) = 4^3$$
$$f(4) = 16$$

Answer: 16

9. Consider a triangle with sides of lengths 7, 8, and 9. A circle with radius 1 rolls along the triangle's interior, always tangent to at least one side as it rolls along. Determine the length of the path traced by the center of the circle as it completes one full revolution around the triangle's interior. Rationalize your final answer.

Solution. Consider a triangle $\triangle ABC$ with side lengths $BA = 7$, $AC = 8$, and $BC = 9$, as shown in the figure below. The path traced by the center of the circle forms a smaller triangle $\triangle A'B'C'$ inside $\triangle ABC$. Since the corresponding angles of $\triangle A'B'C'$ and $\triangle ABC$ are congruent, the two triangles are similar. Let the side lengths of $\triangle A'B'C'$ be denoted by $7x$, $8x$, and $9x$, where x represents the scaling factor between the two triangles.



To find x, we use the area of the triangle. On one hand, Heron's formula, which one can show using the Pythagorean theorem, yields

$$s = \frac{7+8+9}{2} = 12, \quad \text{Area} = \sqrt{12(12-7)(12-8)(12-9)} = \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = 12\sqrt{5}$$

The triangle consists of three trapezoids and a smaller triangle $\triangle A'B'C'$. With the similarity ratio $1 : x$, we have

$$12\sqrt{5} = \frac{1}{2}\Big[(7+7x) + (8+8x) + (9+9x)\Big] + 12\sqrt{5}x^2,$$

which implies

$$12\sqrt{5}x^2 + 12x + 12(1-\sqrt{5}) = 0, \quad \text{or} \quad \sqrt{5}x^2 + x + (1-\sqrt{5}) = 0.$$

We have

$$x = \frac{-1 \pm \sqrt{1 - 4\sqrt{5}(1-\sqrt{5})}}{2\sqrt{5}} = \frac{-1 \pm \sqrt{21 - 4\sqrt{5}}}{2\sqrt{5}} = \frac{-1 \pm (\sqrt{20} - 1)}{2\sqrt{5}}$$

By rationalizing the positive root, we have

$$x = \frac{\sqrt{20} - 2}{2\sqrt{5}} = \frac{\sqrt{100} - 2\sqrt{5}}{10} = \frac{5 - \sqrt{5}}{5}$$

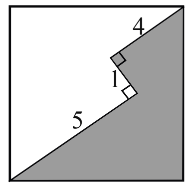
The length of the path is

$$7x + 8x + 9x = 24x = 24\left(\frac{5-\sqrt{5}}{5}\right) = \frac{120 - 24\sqrt{5}}{5}$$

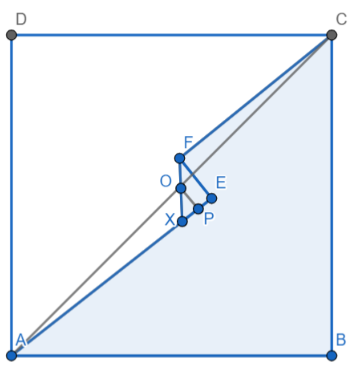
Alternate Solution: The perimeter of $\triangle A'B'C'$ is equal to the perimeter of $\triangle ABC$ minus the perimeter of the triangle similar to $\triangle ABC$ with an inscribed circle of radius 1. As stated above, the area of $\triangle ABC = 12\sqrt{5}$ and the semiperimeter is 12, so the radius of the circle inscribed in $\triangle ABC = \sqrt{5}$. Therefore, the perimeter of $\triangle A'B'C' = 24 - \dfrac{24}{\sqrt{5}}$, which rationalizes to the previous answer.

Answer. $\dfrac{120 - 24\sqrt{5}}{5}$

10. Inside a square, a pair of opposite sides is connected by three line segments with lengths 5, 1, and 4, in that order, as shown below. Find the area of the shaded region.



Solution. Label the points of the square A, B, C, and D, and label points E and F as shown in the figure below. Choose point X on $\overline{AE}$ such that $AX = 4$ (so $EX = 1$). By rotational symmetry, the area of $\triangle XEF$ (which is $\frac{1}{2}$) plus the original shaded area is half the area of square $ABCD$. To find the area of the square, note that the diagonal $\overline{AC}$ bisects $\overline{FX}$ at point we will call O. Draw a line segment $\overline{OP}$ perpendicular to $\overline{EX}$. Then $\triangle APO$ is a right triangle with legs $AP = \frac{9}{2}$ and $OP = \frac{1}{2}$, so $AO = \frac{\sqrt{82}}{2}$ and $AC = 2AO = \sqrt{82}$. Then the area of the square is $\frac{1}{2}(\sqrt{82})^2 = 41$, so our original shaded area is $\frac{1}{2}(41) - \frac{1}{2} = 20$.



Answer 20.

11. Let p be the greatest prime factor of 9991. Compute the sum of the digits of p.

Solution: $9991 = 10000 - 9 = 100^2 - 3^2 = (100-3)(100+3) = 97 \cdot 103$. Hence $p = 103$ and the answer is $1 + 0 + 3 = 4$.

Answer: 4

12. In how many ways can you draw four diagonals inside a convex heptagon, without intersecting each other, to divide it into five triangles, such that each triangle shares at least one side with the heptagon?

Solution. Let $C(n)$ denote the number of ways to draw $n-3$ non-intersecting diagonals inside a convex n-gon, dividing it into $n-2$ triangles. We first show $C(7) = 42$.

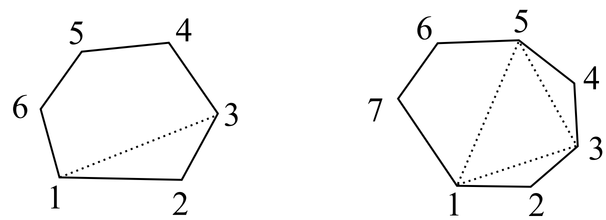
For smaller polygons, we have $C(4) = 2$ and $C(5) = 5$.

To find $C(6)$, consider a hexagon with vertices $P_1, P_2, \ldots, P_6$. The edge P_1P_2 can form a triangle with P_3, P_4, P_5, or P_6. For each case, we calculate the number of ways to divide the remaining polygon:

- With $\triangle P_1P_2P_3$, there are $C(5) = 5$ ways to divide the remaining pentagon.
- With $\triangle P_1P_2P_4$, there are $C(4) = 2$ ways to divide the remaining quadrilateral.
- Similarly, for $\triangle P_1P_2P_5$ and $\triangle P_1P_2P_6$, we have 2 and 5 ways, respectively.

Thus,

$$C(6) = 5 + 2 + 2 + 5 = 14.$$



By a similar process, we compute:

$$C(7) = C(6) + C(5) + 2C(4) + C(5) + C(6) = 42.$$

We exclude triangulations where a triangle does not share a side with the heptagon. There are 7 such cases:

$$\triangle P_1P_3P_5, \triangle P_1P_3P_6, \triangle P_1P_4P_6, \triangle P_2P_4P_6, \triangle P_2P_4P_7, \triangle P_2P_5P_7, \text{ and } \triangle P_3P_5P_7.$$

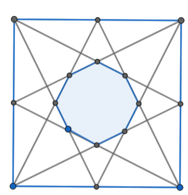
Each of these cases results in 2 possible triangulations of the remaining quadrilateral (see second figure above). Therefore, the number of invalid triangulations is $7 \times 2 = 14$.

Thus, the number of valid triangulations is:

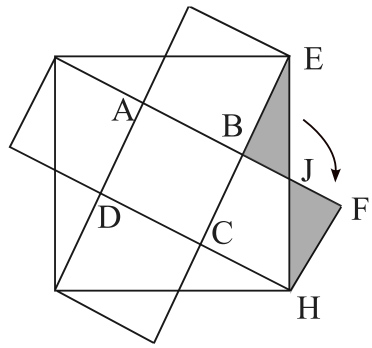
$$C(7) - 14 = 42 - 14 = 28.$$

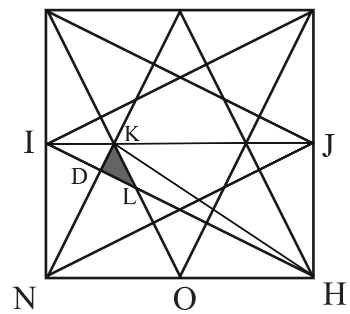
Answer: 28

13. From each vertex of a square with side length 1, draw a line segment to the midpoint of the opposite side's adjacent edges, as shown in the figure. Find the area of the shaded octagon formed by these 8 line segments.



Solution: Consider the smaller square with vertices A, B, C, and D, containing the octagon, as shown in the left figure below. We first demonstrate that the area of this smaller square is $\frac{1}{5}$. To do this, extend the four line segments to create four additional smaller squares. Notice four pairs of congruent triangles, such as $\triangle JBE$ and $\triangle JFH$. By using these congruences, we can express the area of the original square as the sum of the areas of the five smaller squares. Therefore, the area of square $\square ABCD$ is $\frac{1}{5}$.





The area of the octagon is the area of $\square ABCD$ minus four times the shaded triangle in the second figure. To find this area, draw line segments IJ and KH and utilize ratios of similarity. It's immediate to see

$$IK = \frac{1}{2} \times NO = \frac{1}{4},$$

and so the area of $\triangle IKH$ becomes

$$\frac{1}{2} \times IK \times JH = \frac{1}{16}.$$

To this end, we want to know the ratio $DL : IH$. First, from the similarity between $\triangle DKI$ and $\triangle DNH$, we have

$$ID : DH = IK : NH = 1 : 4 \quad \text{or} \quad ID = \frac{1}{5} IH$$

From the similarity between $\triangle LKI$ and $\triangle LOH$, we have

$$IL : LH = IK : OH = 1 : 2 \quad \text{or} \quad IL = \frac{1}{3} IH$$

Therefore,

$$DL = \left(\frac{1}{3} - \frac{1}{5}\right) IH = \frac{2}{15} IH,$$

Now, we have

$$\text{Area } (\triangle KLD) \ = \frac{2}{15} \text{Area } (\triangle IKH) \ = \frac{2}{15} \times \frac{1}{16} = \frac{1}{120}$$

The area of the octagon is

$$\text{Area } (\square ABCD) \ - 4 \times \text{Area } (\triangle KLD) \ = \frac{1}{5} - \frac{1}{30} = \frac{1}{6}$$

Answer: $\frac{1}{6}$

14. Let A, B, and C be three distinct points on the graph of $y = x^2$, with $\overline{AB}$ parallel to the x-axis and $\triangle ABC$ a right triangle with area 2024. What is the sum of the digits of the y-coordinate of C?

Solution: Let (m, m^2) be the coordinates of B. Since $\overline{AB}$ is parallel to the x-axis, the coordinates of A are $(-m, m^2)$. Let the coordinates of C be (n, n^2). $\angle A$ (or $\angle B$) cannot be the

right angle of $\triangle ABC$ since that would make $\overline{AC}$ (or $\overline{BC}$) a vertical line which cannot have two distinct points on the graph of the function.
So $\angle C$ must be the right angle, making $\overline{AC} \perp \overline{BC}$. This means the slopes of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are negative reciprocals, so we have $\dfrac{m^2-n^2}{m-n} = \dfrac{-1(m+n)}{n^2-m^2} \to m^2-n^2 = 1$. Further, the area of $\triangle ABC$ is $\dfrac{1}{2}(2m)(m^2-n^2) = m = 2024$, meaning $n^2 = 2024^2 - 1 = (2000+24)^2 - 1 = 2000^2 + 2(2000)(24) + (24^2 - 1)$. Since $24^2 - 1 < 1000$ and $(2)(2000)(24) < 10^6$, the sum of the digits of n^2 is the sum of the digits of 2000^2, $2(2000)(24)$, and $24^2 - 1$:

$$4 + (9+6) + (5+7+5) = 36$$

Answer: 36

15. Let a, b, c and d be real numbers. Let $P(x) = ax^9 + bx^5 + cx^3 + dx + 8$. Suppose $P(-3) = 9$. Find $P(3)$.

Solution:

$$\begin{gathered} 9 = P(-3) = -a\left(3^9\right) - b\left(3^5\right) - c\left(3^3\right) - d\,(3) + 8 \\ a\left(3^9\right) + b\left(3^5\right) + c\left(3^3\right) + d\,(3) = -1 \\ a\left(3^9\right) + b\left(3^5\right) + c\left(3^3\right) + d\,(3) + 8 = 7 \\ P(3) = 7 \end{gathered}$$

Answer: 7

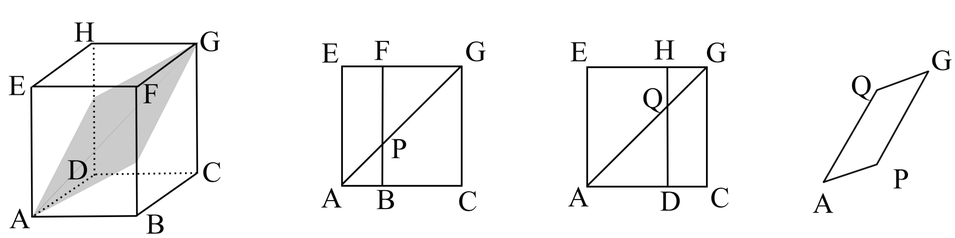
16. Given $1001 = abc + ab + ac + bc + a + b + c + 1$, where a, b, c are positive integers. Compute abc.

Solution: $1001 = abc + ab + ac + bc + a + b + c + 1 = (a+1)(b+1)(c+1)$. Since the prime factorization of 1001 is $7 \cdot 11 \cdot 13$, we deduce $\{a, b, c\} = \{6, 10, 12\}$, and hence $abc = 720$.

Answer: 720

17. Consider a rectangular box with side lengths 1, 2, and 3. A plane cuts through the box, passing through the two opposite vertices A and G and containing a shortest path between these vertices on the box's surface. Find the area of the cross-section formed by this plane.

Solution. Consider a rectangular box with side lengths 1, 2, and 3. Let the vertices of the box be labeled $A, B, \ldots, H$ as shown in the figure. Assume without loss of generality that $AB = 1$, $BC = 2$, and $CG = 3$.



The first planar figure shows a shortest path on the box's surface passing along edge FB, giving $AP = \sqrt{2}$ and $PG = 2\sqrt{2}$. Similarly, another shortest path passing along edge $\overline{HD}$ gives $GQ = \sqrt{2}$ and $QA = 2\sqrt{2}$. (NOTE that placing points P and Q on $\overline{EF}$ and $\overline{CD}$ in a minimal way as above gives us a longer path of length $5\sqrt{2}$. Similarly, placing the points on $\overline{BC}$ and $\overline{EH}$ yields a path of length $4\sqrt{2}$).

To find the area of the cross-section, we calculate the diagonal of the box joining A and G. Applying the Pythagorean theorem twice–first on $\triangle ABC$ and then on $\triangle ACG$–we find:

$$AG = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

The cross-section forms a quadrilateral consisting of two triangles, each with sides $\sqrt{2}$, $2\sqrt{2}$, and $\sqrt{14}$. Using Heron's formula, we calculate the area of one of these triangles. The semi-perimeter s is:

$$s = \frac{\sqrt{2} + 2\sqrt{2} + \sqrt{14}}{2}.$$

The area α of one triangle is:

$$\alpha = \sqrt{s(s - \sqrt{2})(s - 2\sqrt{2})(s - \sqrt{14})}.$$

Now, simplifying the product of terms:

$$\alpha^2 = \frac{(3\sqrt{2} + \sqrt{14})(3\sqrt{2} - \sqrt{14})(\sqrt{2} + \sqrt{14})(\sqrt{14} - \sqrt{2})}{16} = \frac{(18 - 14)(14 - 2)}{16} = \frac{48}{16} = 3.$$

Thus, $\alpha = \sqrt{3}$.

Therefore, the total area of the cross-section is:

$$2\alpha = 2\sqrt{3}.$$

Answer: $2\sqrt{3}$

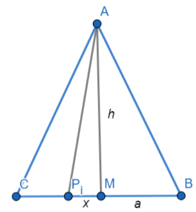
18. Let c, d, and e be real numbers such that the polynomial $x^4 - 26x^3 + cx^2 + dx + e$ has four roots which are consecutive integers. What is e?

Solution. If the smallest of the four roots is r, then the coefficient of x^3 is the opposite of the sum of the roots, so we have $r + (r+1) + (r+2) + (r+3) = 26$. Hence, $4r + 6 = 26$, and $r = 5$. So, e is the product of the four integers: $5 \cdot 6 \cdot 7 \cdot 8 = 1680$.

Answer: 1680

19. Consider points $P_1, P_2, \ldots, P_{100}$ on side $\overline{BC}$ of an isosceles triangle $\triangle ABC$ with $AB = AC = 2$. For each point P_i, define $k_i = AP_i^2 + BP_i \times P_iC$ for $i = 1, 2, \ldots, 100$. Find the sum $k_1 + k_2 + \cdots + k_{100}$

Solution. Draw the height of the triangle $\overline{AM}$, which bisects $\overline{BC}$. Let $MP_i = x$, $AM = h$, and $BM = CM = a$. Then $AP_i^2 = h^2 + x^2$, $BP_i \times P_iC = (a + x)(a - x) = a^2 - x^2$. So $k_i = \left(h^2 + x^2\right) + \left(a^2 - x^2\right) = h^2 + a^2 = AB^2 = 4$.

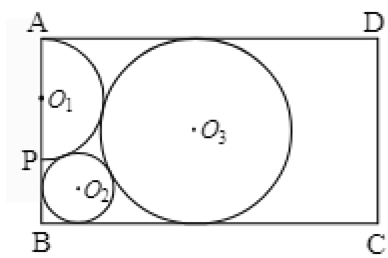


This holds for every $i = 1, 2, \cdots, 100$. Now, the sum is

$$k_1 + k_2 + \cdots + k_{100} = 4 \times 100 = 400.$$

Answer: 400

20. Inside rectangle $ABCD$ with $AB = 1$, a semicircle O_1 is tangent to side $\overline{AD}$ and to two other circles, O_2 and O_3, as shown in the figure. Circle O_2 is tangent to sides $\overline{AB}$ and $\overline{BC}$, as well as to circle O_3. Circle O_3 is tangent to sides $\overline{BC}$ and $\overline{AC}$. Given that $\overline{AP}$ is the diameter of semicircle O_1, find the radius of O_2.

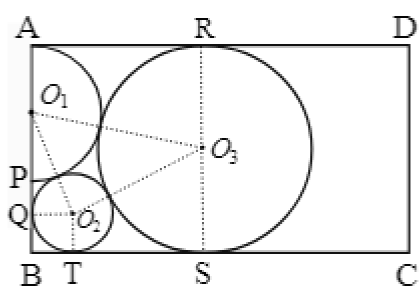


Solution: Let x and y denote the radii of circles O_1 and O_2, respectively. As shown in the figure below, at the tangent point Q between circle O_2 and $\overleftrightarrow{PB}$, we apply the Pythagorean theorem to the right triangle with hypotenuse $\overline{O_1O_2}$:

$$O_1Q = \sqrt{(x+y)^2 - y^2} = \sqrt{x^2 + 2xy},$$

which leads to:

$$AB = AO_1 + O_1Q + QB = x + \sqrt{x^2 + 2xy} + y = 1.$$



For the second equation, using the fact that $\overline{AR}$ is tangent to both circles O_1 and O_3:

$$AR = 2\sqrt{AO_1 \times RO_3} = 2\sqrt{x \times \frac{1}{2}} = \sqrt{2x}.$$

Similarly,

$$TS = 2\sqrt{O_2T \times O_3S} = 2\sqrt{y \times \frac{1}{2}} = \sqrt{2y}.$$

Thus, we obtain the second equation:

$$AR = BT + TS \quad \Rightarrow \quad \sqrt{2x} = y + \sqrt{2y}.$$

To solve the system, we rewrite the first equation:

$$x + \sqrt{x^2 + 2xy} + y = 1 \quad \Rightarrow \quad x^2 + 2xy = (1 - x - y)^2 \quad \Rightarrow \quad 2x = (1 - y)^2.$$

Substituting into the second equation:

$$\sqrt{2x} = y + \sqrt{2y} \quad \Rightarrow \quad 1 - y = y + \sqrt{2y} \quad \Rightarrow \quad 4y^2 - 6y + 1 = 0.$$

Since y must be less than 1, we take $y = \dfrac{3 - \sqrt{5}}{4}$.

Answer: $\dfrac{3 - \sqrt{5}}{4}$