Solutions to DE Exam

Texas A&M High School Math Contest
2 November, 2024

1. Let $X = \log(2000) + \log(2001) + \log(2002) + \cdots + \log(2024)$ and let $Y = \log(100/2000) + \log(100/2001) + \log(100/2002) + \cdots + \log(100/2024)$. Compute $X + Y$.

   Solution:

   $$X + Y = (\log(2000) + \log(100/2000)) + (\log(2001) + \log(100/2001))$$
   $$+ \cdots + (\log(2024) + \log(100/2024)) = 25\log(100) = 50$$

   Answer: 50

2. Evaluate $1 - 2 + 3 - 4 + 5 - 6 + \cdots + 2023 - 2024$.

   Solution:

   $1-2+3-4+5-6+...+2023-2024 = (1-2)+(3-4)+(5-6)+...+(2023-2024) = (-1)(1012) = -1012$

   Answer: -1012

3. The function f has the property that, for each real number x in its domain,

   $$f(x) + f\left(\frac{1}{x}\right) = x$$

   What is the largest possible domain of f?

   Solution. Let a be any real number in the domain of f. Then we have

   $$f(a) + f\left(\frac{1}{a}\right) = f\left(\frac{1}{a}\right) + f(a)$$
   $$a = \frac{1}{a}$$

   So the only possible values of a are ± 1.

   Answer: $\{-1, 1\}$

4. Find a real number b such that the equation $|x + 20| + |x + 3| = b$ has infinitely many solutions.

   Solution: $|x + 20| + |x + 3|$ equals the sum of the distances from the point x to -20 and -3 on the real number line. For $-20 \le x \le -3$, the sum is 17. Therefore $b = 17$.

   Answer: 17

5. What is the smallest positive integer n such that n is divisible by 24, n^2 is a perfect cube, and n^3 is a perfect square? You may leave your answer in the form of an exponent.

   Solution: We know that $n^2 = k^3$ for some positive integer k, and $n^3 = m^2$ for some positive integer m. This means that

   $$n^6 = k^9 \text{ and } n^6 = m^4$$

   To satisfy both equations, $n^6 = a^{36}$ for some positive integer a, so $n = a^6$. Since n is divisible by 24, a is divisible by 2 and 3, the prime factors of 24, so the smallest possible value of a is 6. $n = 6^6$.

   Answer: 6^6 or 36^3 or 216^2 or 46656

6. Suppose that $(3^a)^b = 3^a \times 3^b$. If $b = 5$, find a.

   Solution: We have

   $$\begin{aligned} (3^a)^b &= 3^a \times 3^b \\ ab &= a + b \\ 5a &= a + 5 \\ a &= \frac{5}{4} \end{aligned}$$

   Answer: $\dfrac{5}{4}$

7. Given $\tan(\theta^\circ) = \dfrac{\cos(24^\circ) - \sin(24^\circ)}{\cos(24^\circ) + \sin(24^\circ)}$, what is the smallest positive degree value of θ?

   Solution:

   $$\begin{aligned} \frac{\cos(24^\circ) - \sin(24^\circ)}{\cos(24^\circ) + \sin(24^\circ)} &= \frac{\sin(66^\circ) - \sin(24^\circ)}{\sin(66^\circ) + \sin(66^\circ)} \\ &= \frac{\sin(45^\circ + 21^\circ) - \sin(45^\circ - 21^\circ)}{\sin(45^\circ + 21^\circ) + \sin(45^\circ - 21^\circ)} \\ &= \frac{2\cos(45^\circ)\sin(21^\circ)}{2\sin(45^\circ)\cos(21^\circ)} = \tan(21^\circ) \end{aligned}$$

   So $\theta = 21$.

   Answer: 21

8. Solve the equation

   $$\log_3(x) + \log_3(x^2 - 3x + 3) = 0$$

   Solution: We have

   $$\begin{aligned} \log_3(x(x^2 - 3x + 3)) &= 0 \\ x(x^2 - 3x + 3) &= 1 \\ x^3 - 3x^2 + 3x - 1 &= 0 \\ (x - 1)^3 &= 0 \\ x &= 1 \end{aligned}$$

   Answer: 1

9. For any real number x, we have $3f(x) + f(8 - x) = x^3$. Find $f(4)$

Solution: We have

$$3f(4) + f(4) = 4^3$$
$$4f(4) = 4^3$$
$$f(4) = 16$$

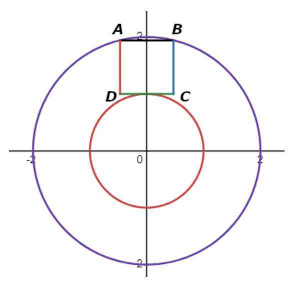
Answer: 16

10. Given the circles centered at the origin with radius 1 and 2. Create square $ABCD$ such that:

- $\overline{CD}$ is tangent to the smaller circle at its midpoint (0, 1), and
- points A and B lie on the larger circle above $\overline{CD}$.

What is the side length of the square?

Solution: The figure is drawn below:



Without loss of generality, suppose point B is in quadrant one with coordinates $(x, \sqrt{4 - x^2})$. By symmetry, the coordinates of A are $(-x, \sqrt{4 - x^2})$. Therefore $AB = 2x$ and $BC = \sqrt{4 - x^2} - 1$, so we must have

$$2x = \sqrt{4 - x^2} - 1$$
$$4x^2 + 4x + 1 = 4 - x^2$$
$$5x^2 + 4x - 3 = 0$$

By the quadratic formula (and taking only the positive square root to yield a positive x value),

$$x = \frac{-4 + \sqrt{16 + 60}}{10} = \frac{-2 + \sqrt{19}}{5}$$

So the side length of the square is $2x = \dfrac{-4 + 2\sqrt{19}}{5}$.

Answer: $\dfrac{-4 + 2\sqrt{19}}{5}$

11. Ten thousand light bulbs, numbered 1 to 10000, are all initially turned on. Each bulb has a switch that toggles its state (turning it on if it's off, or off if it's on). Person 1 toggles the switch on bulb #2 (the first prime number), and every multiple of 2. Person 2 toggles the switch on bulb #3 (the second prime number) and every multiple of 3. Person 3 toggles the switch on bulb #5 (the third prime number) and every multiple of 5. The pattern continues, with Person N toggling the switch on the bulb numbered with the Nth prime number and every multiple of

that prime number. After all the people have completed their turns, which of the following bulbs will be on? If none, write **NONE**:

$$24,\ 112,\ 2024,\ 2025$$

Solution: A bulb's switch is toggled by person N if and only if the Nth prime number is a factor of the bulb number (regardless of multiplicity). Since a switch must be toggled an even number of times to be on at the end of the process, only bulb numbers with an even number of DISTINCT prime factors will remain on:

$$24 = 2^3 \cdot 3 \text{ ON}$$
$$112 = 2^4 \cdot 7 \text{ ON}$$
$$2024 = 2^3 \cdot 11 \cdot 23 \text{ OFF}$$
$$2025 = 3^4 \cdot 5^2 \text{ ON}$$

Answer: 24, 112, 2025

12. Let p be the greatest prime factor of 9991. Compute the sum of the digits of p.

Solution: $9991 = 10000 - 9 = 100^2 - 3^2 = (100 - 3)(100 + 3) = 97 \cdot 103$. Hence $p = 103$ and the answer is $1 + 0 + 3 = 4$.

Answer: 4

13. Let A, B, and C be three distinct points on the graph of $y = x^2$, with $\overline{AB}$ parallel to the x-axis and $\triangle ABC$ a right triangle with area 2024. What is the sum of the digits of the y-coordinate of C?

Solution: Let $(m,\, m^2)$ be the coordinates of B. Since $\overline{AB}$ is parallel to the x-axis, the coordinates of A are $(-m,\, m^2)$. Let the coordinates of C be $(n,\, n^2)$. $\angle A$ (or $\angle B$) cannot be right angle of $\triangle ABC$ since that would make $\overline{AC}$ (or $\overline{BC}$) a vertical line which cannot have two distinct points on the graph of the function.
So $\angle C$ must be the right angle, making $\overline{AC} \perp \overline{BC}$. This means the slopes of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are negative reciprocals, so we have $\dfrac{m^2 - n^2}{m - n} = \dfrac{-1(m + n)}{n^2 - m^2} \to m^2 - n^2 = 1$. Further, the area of $\triangle ABC$ is $\dfrac{1}{2}(2m)(m^2 - n^2) = m = 2024$, meaning $n^2 = 2024^2 - 1 = (2000 + 24)^2 - 1 = 2000^2 + 2(2000)(24) + (24^2 - 1)$. Since $24^2 - 1 < 1000$ and $(2)(2000)(24) < 10^6$, the sum of the digits of n^2 is the sum of the digits of 2000^2, $2(2000)(24)$, and $24^2 - 1$:

$$4 + (9 + 6) + (5 + 7 + 5) = 36$$

Answer: 36

14. In the figures below, each tiny square is one square unit of area (so the area of Figure 0 is 1, and so on). If we continue to follow the pattern, what is the area of Figure 200?

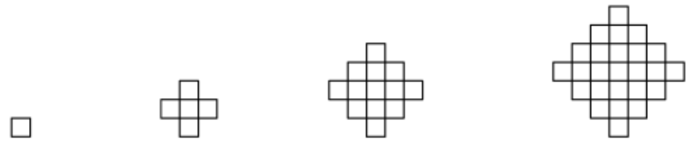


Figure 0   Figure 1   Figure 2   Figure 3

Solution: Notice that Figure n is formed by attaching n squares along each diagonal as shown in the figure below, thus adding a total area of $4n$ square units.

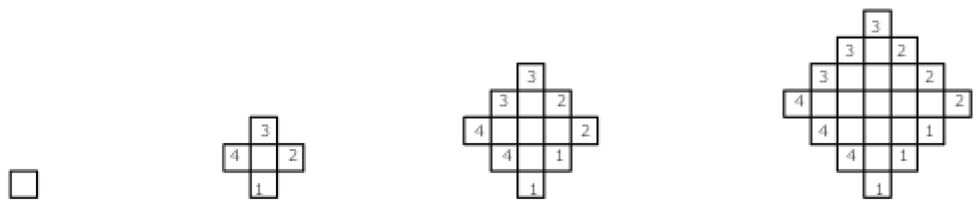


Figure 0   Figure 1   Figure 2   Figure 3

The area of Figure 200 is therefore

$$1 + 4 + 8 + \cdots + 4(200)$$
$$= 1 + 4(1 + 2 + \cdots 200)$$
$$= 1 + 4\left(\frac{(200)(201)}{2}\right) = 80,401$$

Answer: 80, 401

15. Let a, b, c and d be real numbers. Let $P(x) = ax^9 + bx^5 + cx^3 + dx + 8$. Suppose $P(-3) = 9$. Find $P(3)$.

Solution:

$$9 = P(-3) = -a\left(3^9\right) - b\left(3^5\right) - c\left(3^3\right) - d\,(3) + 8$$
$$a\left(3^9\right) + b\left(3^5\right) + c\left(3^3\right) + d\,(3) = -1$$
$$a\left(3^9\right) + b\left(3^5\right) + c\left(3^3\right) + d\,(3) + 8 = 7$$
$$P(3) = 7$$

Answer: 7

16. Given $1001 = abc + ab + ac + bc + a + b + c + 1$, where a, b, c are positive integers. Compute abc..

    Solution: $1001 = abc + ab + ac + bc + a + b + c + 1 = (a + 1)(b + 1)(c + 1)$. Since the prime factorization of 1001 is $7 \cdot 11 \cdot 13$, we deduce $\{a, b, c\} = \{6, 10, 12\}$, and hence $abc = 720$.

    Answer: 720

17. Find the number of real ordered pairs $(a,\, b)$ which are solutions to the complex equation $(a + bi)^{2024} = a - bi$.

    Solution: If $s = \sqrt{a^2 + b^2} = |a + bi|$, then the magnitude of $(a + bi)^{2024}$ is s^{2024} and the magnitude of $a - bi$ is s. So we must have $s^{2024} = s$, meaning $s = 0$ or $s = 1$. If $s = 0$, then $(0,\, 0)$ is one Solution:
    If $s = 1$, multiply both sides of the equation by $a + bi$:

    $$(a + bi)^{2025} = (a - bi)(a + bi) = a^2 + b^2 = s^2 = 1$$

    Using the polar form of the complex number $a + bi$, we know there are 2025 complex roots of unity, hence, 2025 solutions when $s = 1$. Putting these cases together, we have 2026 solutions.

    Answer: 2026

18. Let c, d, and e be real numbers such that the polynomial $x^4 - 24x^3 + cx^2 + dx + e$ has four distinct roots which are positive integers in arithmetic progression. What is e?

    Solution: Let x, $x + s$, $x + 2s$, and $x + 3s$ be the roots (with $s > 0$). The sum of these roots is the opposite of the coefficient of x^3, so

    $$x + (x + s) + (x + 2s) + (x + 3s) = 24$$

    $$4x + 6s = 24$$

    The only value of s for which x is a positive integer is $s = 2$, so $x = 3$. Therefore, the roots are 3, 5, 7, and 9. e is the product of these roots, so $e = 3 \cdot 5 \cdot 7 \cdot 9 = 945$.

    Answer: 945

19. The square below is a multiplicative magic square, where the product of the numbers in each row, column, and diagonal is the same. If all the entries are positive integers (not necessarily distinct), how many different possible values are there for h?

6	b	c
d	e	f
g	h	24

    Solution: We can write the other entries in terms of h. From the diagonals and center row and column,

    $$144e = ceg = beh = def$$

    Since all entries are positive, we can say

    $$b = \frac{144}{h},\ c = \frac{144}{g},\ \text{and } d = \frac{144}{f}$$

    From the top and bottom rows,

    $$24hg = 6bc = \frac{6 \cdot (144)^2}{hg} \Rightarrow g = \frac{72}{h} \Rightarrow c = 2h$$

From the first and third columns,

$$6dg = 24cf \Rightarrow \frac{6 \cdot 144 \cdot 72}{fh} = 48fh \Rightarrow f = \frac{36}{h} \Rightarrow d = 4h$$

Finally, the first row and main diagonal yields $6bc = 144e \Rightarrow (6)\left(\frac{144}{h}\right)(2h) = 144e$, so $e = 12$.

All of our entries will be positive integers if and only if h is a positive integer and $h|36$. Therefore, the possible values of h are 1, 2, 3, 4, 6, 9, 12, 18, and 36, which is 9 values.

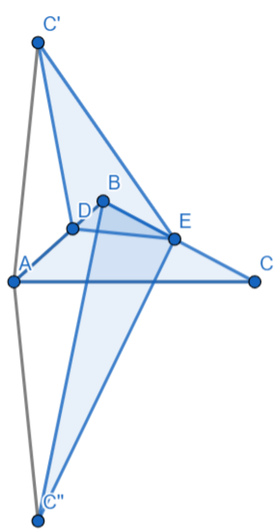
Answer: 9

20. In $\triangle BAC$, $\angle BAC = 40°$, $AB = 2$, and $AC = 4$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively. If the minimum possible value of $BE + DE + CD = \sqrt{N}$, what is N?

Solution: In the figure below, C' is the reflection of point C across $\overline{AB}$. Using the Triangle Inequality on $\triangle EDC'$, we have $ED + DC' \geq EC'$, so by the reflection, $ED + DC \geq EC'$. Now let C" be the reflection of point C' across $\overline{AC}$. Using the Triangle Inequality on $\triangle BEC$", we have $BE + EC" \geq BC"$, so by the reflection, $BE + EC' \geq BC"$, so $BE + ED + DC \geq BC"$, meaning the minimum value occurs at equality.
From the reflections, we know that $AC = AC' = AC" = 2$, $\angle BAC' = \angle BAC = 40°$, and $\angle CAC" = \angle CAC' = \angle BAC' + \angle BAC = 80°$, making $\angle BAC" = 120°$. Using the Law of Cosines on $\triangle BAC$", we have

$$(BC")^2 = AB^2 + (AC")^2 - 2AB \cdot AC" \cos(120°)$$
$$= 2^2 + 4^2 - 2(2)(4)\left(-\frac{1}{2}\right) = 28$$

so $BC" = \sqrt{28}$, making $N = 28$.



Answer: 28