Solutions to EF Exam

Texas A&M High School Math Contest
2 November, 2024

1. A six-digit integer is written down. If the units digit is moved to the front of the number (for example, 123456 becomes 612345), the new integer is five times the original integer. What is the original integer?

   Solution: Algebraically, let x be the value of the first 5 digits of the integer and y be the units digit. The original integer is then $10x + y$, while the new integer is $100000y + x$. So

   $$100000y + x = 5(10x + y)$$
   $$49x = 99995y$$
   $$7x = 14285y$$

   So we must have $x = 14285$ and $y = 7$, making our original integer 142857.
   NOTE that this can also be solved by recognizing the pattern in the repeating decimal of fractions with denominator 7:
   $$\frac{1}{7} = .\overline{142857},\ \frac{5}{7} = .\overline{714285}$$

   Answer: 142857

2. The function f has the property that, for each real number x in its domain,
   $$f(x) + f\left(\frac{1}{x}\right) = x$$
   What is the largest possible domain of f?

   Solution: Let a be any real number in the domain of f. Then we have
   $$f(a) + f\left(\frac{1}{a}\right) = f\left(\frac{1}{a}\right) + f(a)$$
   $$a = \frac{1}{a}$$
   So the only possible values of a are ± 1.

   Answer: $\{-1, 1\}$

3. What is the smallest positive integer n such that n is divisible by 24, n^2 is a perfect cube, and n^3 is a perfect square? You may leave your answer in the form of an exponent.

   Solution: We know that $n^2 = k^3$ for some positive integer k, and $n^3 = m^2$ for some positive integer m. This means that
   $$n^6 = k^9 \text{ and } n^6 = m^4$$
   To satisfy both equations, $n^6 = a^{36}$ for some positive integer a, so $n = a^6$. Since n is divisible by 24, a is divisible by 2 and 3, the prime factors of 24, so the smallest possible value of a is 6. $n = 6^6$.

   Answer: 6^6 or 36^3 or 216^2 or 46656

4. Given $\tan(\theta^\circ) = \dfrac{\cos(24^\circ) - \sin(24^\circ)}{\cos(24^\circ) + \sin(24^\circ)}$, what is the smallest positive degree value of θ?

Solution:

$$\begin{aligned}\frac{\cos(24^\circ) - \sin(24^\circ)}{\cos(24^\circ) + \sin(24^\circ)} &= \frac{\sin(66^\circ) - \sin(24^\circ)}{\sin(66^\circ) + \sin(24^\circ)} \\ &= \frac{\sin(45^\circ + 21^\circ) - \sin(45^\circ - 21^\circ)}{\sin(45^\circ + 21^\circ) + \sin(45^\circ - 21^\circ)} \\ &= \frac{2\cos(45^\circ)\sin(21^\circ)}{2\sin(45^\circ)\cos(21^\circ)} = \tan(21^\circ)\end{aligned}$$

So $\theta = 21$.

Answer: 21

5. Consider the sequence with general term

$$a_n = \ln(1+n) - 2\ln(2+n) + \ln(3+n), \quad n \geq 1.$$

Find $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k$.

Solution: We have that

$$\begin{aligned}\sum_{k=1}^{n} a_k &= (\ln 2 - 2\ln 3 + \ln 4) + (\ln 3 - 2\ln 4 + \ln 5) + \cdots + (\ln n - 2\ln(1+n) + \ln(2+n)) \\ &+ (\ln(1+n) - 2\ln(2+n) + \ln(3+n)) = \ln 2 - \ln 3 - \ln(2+n) + \ln(3+n) \\ &= \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3+n}{2+n}\right).\end{aligned}$$

Since $\lim\limits_{n\to\infty} \ln\left(\dfrac{3+n}{2+n}\right) = \ln 1 = 0$ we get that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k = \ln\left(\dfrac{2}{3}\right)$ or $\ln 2 - \ln 3$.

Answer: $\ln\left(\dfrac{2}{3}\right)$ or $\ln 2 - \ln 3$

6. Evaluate $L = \lim_{x\to-\infty} \dfrac{2e^{3x} - 5e^{-4x}}{4e^{3x} + 3e^{-4x}}$.

   Solution: We divide the numerator and denominator by e^{-4x} and we get

   $$L = \lim_{x\to-\infty} \frac{2e^{3x} - 5e^{-4x}}{4e^{3x} + 3e^{-4x}} = \lim_{x\to-\infty} \frac{2e^{7x} - 5}{4e^{7x} + 3}.$$

   Using the fact that $\lim_{x\to-\infty} e^{7x} = 0$ we obtain that $L = -\dfrac{5}{3}$.

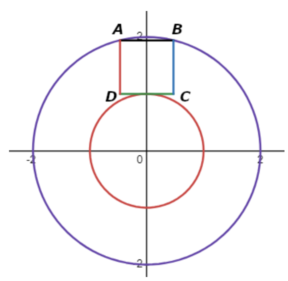
   Answer: $-\dfrac{5}{3}$

7. Given the circles centered at the origin with radius 1 and 2. Create square $ABCD$ such that:

   - $\overline{CD}$ is tangent to the smaller circle at its midpoint (0, 1), and
   - points A and B lie on the larger circle above $\overline{CD}$.

   What is the side length of the square?

   Solution: The figure is drawn below:



   Without loss of generality, suppose point B is in quadrant one with coordinates $(x,\ \sqrt{4-x^2})$. By symmetry, the coordinates of A are $(-x,\ \sqrt{4-x^2})$. Therefore $AB = 2x$ and $BC = \sqrt{4-x^2} - 1$, so we must have

   $$2x = \sqrt{4-x^2} - 1$$
   $$4x^2 + 4x + 1 = 4 - x^2$$
   $$5x^2 + 4x - 3 = 0$$

   By the quadratic formula (and taking only the positive square root to yield a positive x value),

   $$x = \frac{-4 + \sqrt{16+60}}{10} = \frac{-2 + \sqrt{19}}{5}$$

   So the side length of the square is $2x = \dfrac{-4 + 2\sqrt{19}}{5}$.

   Answer: $\dfrac{-4 + 2\sqrt{19}}{5}$

8. Ten thousand light bulbs, numbered 1 to 10000, are all initially turned on. Each bulb has a switch that toggles its state (turning it on if it's off, or off if it's on). Person 1 toggles the switch on bulb #2 (the first prime number), and every multiple of 2. Person 2 toggles the switch on bulb #3 (the second prime number) and every multiple of 3. Person 3 toggles the switch on bulb #5 (the third prime number) and every multiple of 5. The pattern continues, with Person N toggling the switch on the bulb numbered with the Nth prime number and every multiple of that prime number. After all the people have completed their turns, which of the following bulbs will be on? If none, write **NONE**:

$$24,\ 112,\ 2024,\ 2025$$

Solution: A bulb's switch is toggled by person N if and only if the Nth prime number is a factor of the bulb number (regardless of multiplicity). Since a switch must be toggled an even number of times to be on at the end of the process, only bulb numbers with an even number of DISTINCT prime factors will remain on:

$$\begin{gathered} 24 = 2^3 \cdot 3 \text{ ON} \\ 112 = 2^4 \cdot 7 \text{ ON} \\ 2024 = 2^3 \cdot 11 \cdot 23 \text{ OFF} \\ 2025 = 3^4 \cdot 5^2 \text{ ON} \end{gathered}$$

Answer: 24, 112, 2025

9. Find the product ab such that the function

$$f(x) = \begin{cases} xe^x, & x \le 1 \\ ax + b, & x > 1 \end{cases}$$

is differentiable on $(-\infty, \infty)$.

Solution: From the fact that f is continuous at $x = 1$, we get that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \Leftrightarrow e = a + b.$$

Consider the functions $g(x) = xe^x$ and $h(x) = ax + b$ defined on $(-\infty, \infty)$. Their derivatives are $g'(x) = e^x + xe^x$ and $h'(x) = a$ which are continuous on $(-\infty, \infty)$. Since f is differentiable at $x = 1$ we obtain that $g'(1) = h'(1) \Rightarrow a = 2e$. From $a + b = e$ we get that $b = -e$, which implies that $ab = -2e^2$.

Answer: $-2e^2$

10. Let A, B, and C be three distinct points on the graph of $y = x^2$, with $\overline{AB}$ parallel to the x-axis and $\triangle ABC$ a right triangle with area 2024. What is the sum of the digits of the y-coordinate of C?

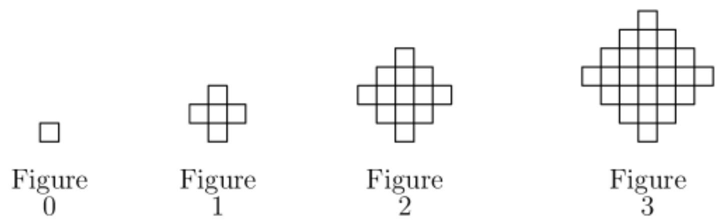
Solution: Let $(m,\, m^2)$ be the coordinates of B. Since $\overline{AB}$ is parallel to the x-axis, the coordinates of A are $(-m,\, m^2)$. Let the coordinates of C be $(n,\, n^2)$. $\angle A$ (or $\angle B$) cannot be right angle of $\triangle ABC$ since that would make $\overline{AC}$ (or $\overline{BC}$) a vertical line which cannot have two distinct points on the graph of the function.
So $\angle C$ must be the right angle, making $\overline{AC} \perp \overline{BC}$. This means the slopes of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are

negative reciprocals, so we have $\frac{m^2 - n^2}{m - n} = \frac{-1(m+n)}{n^2 - m^2} \to m^2 - n^2 = 1$. Further, the area of $\triangle ABC$ is $\frac{1}{2}(2m)(m^2 - n^2) = m = 2024$, meaning $n^2 = 2024^2 - 1 = (2000 + 24)^2 - 1 = 2000^2 + 2(2000)(24) + (24^2 - 1)$. Since $24^2 - 1 < 1000$ and $(2)(2000)(24) < 10^6$, the sum of the digits of n^2 is the sum of the digits of 2000^2, $2(2000)(24)$, and $24^2 - 1$:

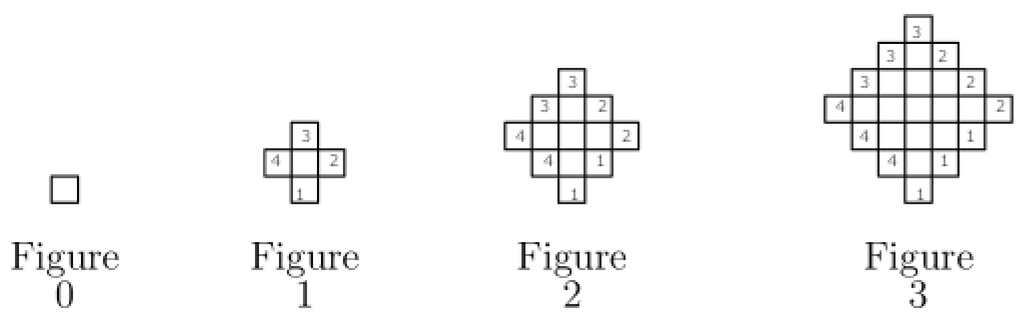
$$4 + (9 + 6) + (5 + 7 + 5) = 36$$

Answer: 36

11. In the figures below, each tiny square is one square unit of area (so the area of Figure 0 is 1, and so on). If we continue to follow the pattern, what is the area of Figure 200?



Solution: Notice that Figure n is formed by attaching n squares along each diagonal as shown in the figure below, thus adding a total area of $4n$ square units.



The area of Figure 200 is therefore

$$1 + 4 + 8 + \cdots + 4(200)$$

$$= 1 + 4(1 + 2 + \cdots 200)$$

$$= 1 + 4\left(\frac{(200)(201)}{2}\right) = 80,401$$

Answer: 80, 401

12. We know that the sequence $\{a_n\}$ defined by

$$a_1 = 1, \quad a_{n+1} = \frac{1}{3 - a_n}$$

is bounded and decreasing. Find $\lim\limits_{n\to\infty} a_n$.

Solution: Since $\{a_n\}$ is bounded and monotonic we get, by using the Monotonic Sequence Theorem, that $\{a_n\}$ is convergent. Let $L = \lim\limits_{n\to\infty} a_n$. From the recurrence relation $a_{n+1} = \dfrac{1}{3 - a_n}$ we obtain that

$$L = \frac{1}{3 - L} \Leftrightarrow L^2 - 3L + 1 = 0 \Leftrightarrow L = \frac{3 - \sqrt{5}}{2} \text{ or } L = \frac{3 + \sqrt{5}}{2}.$$

But $\{a_n\}$ is decreasing and $a_1 = 1$. So $a_n \leq a_1 = 1$ for all positive integers n which implies that $L \leq 1$. Therefore, $L = \dfrac{3 - \sqrt{5}}{2}$.

Answer: $\dfrac{3 - \sqrt{5}}{2}$

13. Evaluate $\lim\limits_{n\to\infty} \left(\dfrac{1}{\sqrt{4n^2 - 1^2}} + \dfrac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \dfrac{1}{\sqrt{4n^2 - n^2}} \right)$.

Solution: We write the above limit as

$$\lim_{n\to\infty} \frac{1}{n} \left(\frac{1}{\sqrt{4 - (\frac{1}{n})^2}} + \frac{1}{\sqrt{4 - (\frac{2}{n})^2}} + \cdots + \frac{1}{\sqrt{4 - (\frac{n}{n})^2}} \right),$$

which implies that

$$L = \int_0^1 \frac{1}{\sqrt{4 - x^2}} dx = \arcsin \frac{x}{2} \Big|_0^1 = \arcsin \frac{1}{2} = \frac{\pi}{6}.$$

Answer: $\dfrac{\pi}{6}$

14. Find the number of real ordered pairs (a, b) which are solutions to the complex equation $(a + bi)^{2024} = a - bi$.

Solution: If $s = \sqrt{a^2 + b^2} = |a + bi|$, then the magnitude of $(a + bi)^{2024}$ is s^{2024} and the magnitude of $a - bi$ is s. So we must have $s^{2024} = s$, meaning $s = 0$ or $s = 1$. If $s = 0$, then $(0, 0)$ is one solution.
If $s = 1$, multiply both sides of the equation by $a + bi$:

$$(a + bi)^{2025} = (a - bi)(a + bi) = a^2 + b^2 = s^2 = 1$$

Using the polar form of the complex number $a + bi$, we know there are 2025 complex roots of unity, hence, 2025 solutions when $s = 1$. Putting these cases together, we have 2026 solutions.

Answer: 2026

15. Let c, d, and e be real numbers such that the polynomial $x^4 - 24x^3 + cx^2 + dx + e$ has four distinct roots which are positive integers in arithmetic progression. What is e?

Solution: Let x, $x+s$, $x+2s$, and $x+3s$ be the roots (with $s > 0$). The sum of these roots is the opposite of the coefficient of x^3, so

$$x + (x+s) + (x+2s) + (x+3s) = 24$$

$$4x + 6s = 24$$

The only value of s for which x is a positive integer is $s = 2$, so $x = 3$. Therefore, the roots are 3, 5, 7, and 9. e is the product of these roots, so $e = 3 \cdot 5 \cdot 7 \cdot 9 = 945$.

Answer: 945

16. Evaluate $\lim\limits_{x\to\infty} \left(\dfrac{x^2+2x+9}{x^2+5x+3} \right)^x$.

Solution: Using $\lim\limits_{u\to 0}(1+u)^{\frac{1}{u}} = e$ we get

$$\begin{aligned}
\lim_{x\to\infty} \left(\frac{x^2+2x+9}{x^2+5x+3} \right)^x &= \lim_{x\to\infty} \left(1 + \frac{x^2+2x+9}{x^2+5x+3} - 1 \right)^x \\
&= \lim_{x\to\infty} \left[\left(1 + \frac{6-3x}{x^2+5x+3} \right)^{\frac{x^2+5x+3}{6-3x}} \right]^{\frac{6x-3x^2}{x^2+5x+3}} \\
&= e^{\lim\limits_{x\to\infty} \frac{6x-3x^2}{x^2+5x+3}} = e^{-3} = \frac{1}{e^3}.
\end{aligned}$$

Answer: $\dfrac{1}{e^3}$ or e^{-3}

17. The square below is a multiplicative magic square, where the product of the numbers in each row, column, and diagonal is the same. If all the entries are positive integers (not necessarily distinct), how many different possible values are there for h?

6	b	c
d	e	f
g	h	24

Solution: We can write the other entries in terms of h. From the diagonals and center row and column,

$$144e = ceg = beh = def$$

Since all entries are positive, we can say

$$b = \frac{144}{h},\ c = \frac{144}{g},\ \text{and } d = \frac{144}{f}$$

From the top and bottom rows,

$$24hg = 6bc = \frac{6 \cdot (144)^2}{hg} \to g = \frac{72}{h} \to c = 2h$$

From the first and third columns,

$$6dg = 24cf \to \frac{6 \cdot 144 \cdot 72}{fh} = 48fh \to f = \frac{36}{h} \to d = 4h$$

Finally, the first row and main diagonal yields $6bc = 144e \Rightarrow (6)\left(\frac{144}{h}\right)(2h) = 144e$, so $e = 12$.

All of our entries will be positive integers if and only if h is a positive integer and $h|36$. Therefore, the possible values of h are 1, 2, 3, 4, 6, 9, 12, 18, and 36, which is 9 values.

Answer: 9

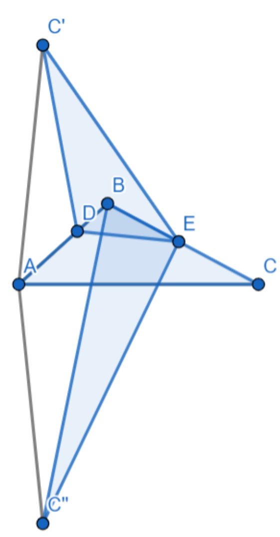
18. In $\triangle BAC$, $\angle BAC = 40^\circ$, $AB = 2$, and $AC = 4$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively. If the minimum possible value of $BE + DE + CD = \sqrt{N}$, what is N?

Solution: In the figure below, C' is the reflection of point C across $\overline{AB}$. Using the Triangle Inequality on $\triangle EDC'$, we have $ED + DC' \geq EC'$, so by the reflection, $ED + DC \geq EC'$. Now let $C"$ be the reflection of point C' across $\overline{AC}$. Using the Triangle Inequality on $\triangle BEC"$, we have $BE + EC" \geq BC"$, so by the reflection, $BE + EC' \geq BC"$, so $BE + ED + DC \geq BC"$, meaning the minimum value occurs at equality.
From the reflections, we know that $AC = AC' = AC" = 2$, $\angle BAC' = \angle BAC = 40^\circ$, and $\angle CAC" = \angle CAC' = \angle BAC' + \angle BAC = 80^\circ$, making $\angle BAC" = 120^\circ$. Using the Law of Cosines on $\triangle BAC"$, we have

$$(BC")^2 = AB^2 + (AC")^2 - 2AB \cdot AC" \cos(120^\circ)$$

$$= 2^2 + 4^2 - 2(2)(4)\left(-\frac{1}{2}\right) = 28$$

so $BC" = \sqrt{28}$, making $N = 28$.



Answer: 28

19. The function $f(x) = x^3 - 3x$ is one-to-one on the interval $(1, \infty)$. Let g be the inverse of f on $(1, \infty)$. Find $g''(2)$.

Solution: From $f(2) = 2$ we get that $g(2) = 2$. We have that

$$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(2)} = \frac{1}{9},$$

since $f'(x) = 3x^2 - 3$. Then from $f(x) = x^3 - 3x$ we can write

$$g^3(y) - 3g(y) = y \Rightarrow 3g^2(y)g'(y) - 3g'(y) = 1$$

and

$$6g(y)(g'(y))^2 + 3g^2(y)g''(y) - 3g''(y) = 0.$$

If we set $y = 2$ we obtain $6 \cdot 2 \cdot \frac{1}{81} + 3 \cdot 4 \cdot g''(2) - 3g''(2) = 0$ which implies that $g''(2) = -\frac{4}{243}$.

Answer: $-\frac{4}{243}$

20. Evaluate the integral

$$I = \int_0^{\frac{\pi}{2}} \frac{2\sin x + 3\cos x}{4\sin x + 5\cos x} dx.$$

Solution: Since $\frac{d}{dx}(4\sin x + 5\cos x) = 4\cos x - 5\sin x$, we find A and B such that

$$2\sin x + 3\cos x = A(4\sin x + 5\cos x) + B(4\cos x - 5\sin x).$$

We obtain the equations $4A - 5B = 2$ and $5A + 4B = 3$ which gives $A = \frac{23}{41}$ and $B = \frac{2}{41}$. Therefore,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{23}{41} dx + \frac{2}{41} \int_0^{\frac{\pi}{2}} \frac{(4\sin x + 5\cos x)'}{4\sin x + 5\cos x} dx \\ &= \frac{23}{41} \cdot \frac{\pi}{2} + \frac{2}{41} \ln |4\sin x + 5\cos x| \Big|_0^{\frac{\pi}{2}} = \frac{23\pi}{82} + \frac{2}{41} \ln\left(\frac{4}{5}\right). \end{aligned}$$

Answer: $\frac{23\pi}{82} + \frac{2}{41} \ln\left(\frac{4}{5}\right)$