Stopping the flood: solutions

A landscape in the Square Country is an infinite checkerboard: each cell is Water, Ground, or Rock.

If the Ground cell is adjacent by side to a Water cell on Day n, then this cell will be flooded overnight, i.e. it will be a Water cell on Day $n+1$. Rock cells stay Rock, and Water cells stay Water.

(1) On Day 1, there is only one Water cell and no Rock cells. Find the formula for the number of Water cells on Day n.

On Day n, the Water cells will occupy the diamond-shaped region, with the longest row being $2n-1$. The area of such a diamond-shaped region is $1+3+\cdots+(2n-3)+(2n-1)+(2n-3)+\cdots+1$. We get $1+3+\cdots+(2n-1) = \frac{(2n)\cdot n}{2} = n^2$ and $(2n-3)+\cdots+1 = (n-1)^2$, thus there will be $n^2+(n-1)^2$ Water cells on Day n.

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(2) Suppose there are no Rock cells. Which configurations are possible Water configurations on Day 2? Describe a criterion to distinguish possible configurations from impossible configurations.

We will say that a cell is a Day-1 Water cell if it existed on Day 1, and a Day-2 Water cell if it was filled with Water on Day 2.

All possible configurations satisfy the following property:

If a Water cell is adjacent to a Ground cell, then one of its neighbors is a Water cell that is completely surrounded by Water

Indeed, all Day-1 Water cells are completely surrounded by water. Thus all Water cells adjacent to Ground cells are Day-2 Water cells. They appeared on Day 2 because they were adjacent to some Day-1 Water cell.

Thus a Water cell that is adjacent to the Ground cell has a Water neighbor that is completely surrounded by Water.

Prove that if a configuration satisfies this property, then it is possible. Indeed, on Day 1, put water in all cells that are completely surrounded by water in our configuration. Then on Day 2, all cells of our configuration will be filled in: cells surrounded by water appear on Day 1, cells adjacent to the Ground will be filled in on Day 2 due to our property. Also, no extra cells will fill in with water since on Day 1, we do not put water in the cells that are adjacent to the Ground in our configuration.

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(3) We will say that *the flood has stopped* on Day k if Water does not fill any new cells after Day k. Prove that if the flood has

stopped and Water occupies m^2 cells, then there are at least $2m$ Rock cells in the landscape.

Since Water occupies m^2 cells, our configuration of Water cells either spans at least m rows, or at least m columns. Indeed, if we have at most $m-1$ columns and $m-1$ rows, then we have at most $(m-1)^2$ cells.

Suppose that our configuration spans at least m rows. In each row, there must be at least 2 Rock cells: one on the left end and one on the right end. Otherwise the flood will continue to the left or to the right.

Thus we have at least $2m$ Rock cells.

(4) If the flood has stopped and Rock occupies at most N cells, determine the largest possible number of Water cells. Prove your answer.

Answer: If $N = 4n$, we have at most $n^2 + (n-1)^2$ Water cells; for $N = 4n+1$, at most $2n^2 - n$ Water cells; for $N = 4n+2$, at most $2n^2$ Water cells; for $N = 4n+3$, at most $2n^2 + n$ Water cells.

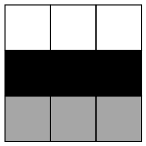
For any N, the maximal number of Water cells equals $\lfloor \frac{N^2-4N+8}{8} \rfloor$.

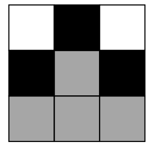
Solution: Consider an optimal configuration: the one with the maximal possible number of Water cells among all configurations with at most N Rock cells. If this configuration has islands (Rock inside Water), we can replace Rocks by Water and the configuration is not optimal. If we have several lakes, we can move the lakes closer together without changing their shapes until they are next to each other (i.e. share the interval on the boundary). This removes some Rock cells and does not change the number of Water cells, and also unites several lakes into one. Below we will assume that our configuration is one lake without islands. In other words, the shoreline is a single polygonal line with angles $90°$, $180°$, and $270°$.

Walking along the shoreline and keeping Water on your right, we will write down our moves as Right, Left, Up, or Down; R, L, U, D for short. Clearly, the sequence cannot contain UD, DU, RL, LR.

No straight segments of length 3.

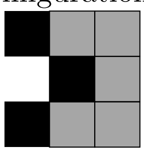
If the sequence contains RRR, it can be improved as shown below (replaced by RURDR) which increases the number of Water cells by 1 and does not change the number of Rock cells. Similar transformation works for UUU, DDD, LLL. Hence these combinations do not appear for the optimal configuration.

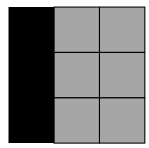




No backtracking.

If the shoreline contains RUL, this can be replaced with U. The number of Water cells increases by 1 while the number of Rock cells does not change. Similarly, RUUL can be replaced with UU, thus RUL and RUUL cannot appear for the optimal configuration.





Roughly rectangular-shaped shoreline.

Assume without loss of generality that the sequence starts with RU, followed by a certain amount of R and U. We will call this an RU-piece of the shoreline.

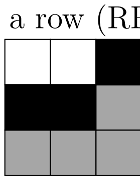
Due to the above, the only way this RU-piece may end is URD or URRD, since RUL and RUUL is impossible, and UUU or RRR are also impossible.

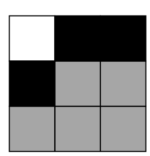
Then, there will be a certain amount of R, D; we will call this an RD-piece of a shoreline. Due to similar arguments, the only way the RD-piece may end is either RDL or RDDL.

Proceeding in a similar way, we will split the shoreline into four pieces: RU-, RD-, LD-, and LU-piece; after that we come back to the initial RU-piece. Formally speaking, we may have 8, 12, or more pieces (RU-RD-LD-LU-RU-RD-LD-LU etc) in the shoreline. However, since the shoreline may only make one turn around the lake, and every four pieces (RU-RD-LD-LU) constitute a 360-degree turn in a clockwise direction, we will only have four pieces of this form.

Sides of rectangles are diagonal.

Take the RU piece. Recall that we cannot have three letters in a row (RRR or UUU).





If the shoreline contains RRUR, we can increase the number of Water cells and keep same the number of Rock cells by replacing it with RURR. Thus the letter repetition RR can only appear at the end of the RU piece in the optimal configuration. Similarly, the repetition UU may only appear in the beginning. Analogous arguments apply to RD, LD, and LU pieces.

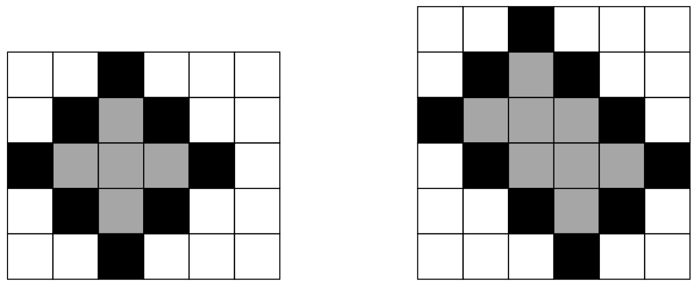


FIGURE 1. Case (1): $n = m$ and $m = n + 1$ for $n = 2$

Analyzing rectangles with diagonal sides.

We have obtained the following possible shapes of the optimal shoreline. All these shapes look like rectangles with four ladder-shaped sides: namely RU, RD, DL, LU sides. The sequence of n pairs RU will be denoted $(RU)_n$. We have the following cases.

(1) No letter repetitions:

$$U(RU)_{n-1}R(DR)_{m-1}D(LD)_{k-1}L(UL)_{l-1}.$$

Since the number of U and D, as well as the number of R and L, must be the same, we have $n+l = m+k$ and $n+m = k+l$ and thus $n = k$ and $m = l$.

If $m = n$ (diamond-shaped configuration), we have $4n$ Rock cells and $n^2 + (n-1)^2$ Water cells, see problem (1).

If $m = n + 1$, we can take all the Rock cells that are adjacent to the piece $D(LD)_{n-1}L(UL)_{m-1}$ (the bottom half of the picture) and move them left. This does not change the number of Water cells, but reduces this case to case (2) below.

If $m - n \geq 2$, this shape is not optimal, since the same movement of Rock cells will add $m - n - 1 > 0$ Water cells and will not change the number of Rock cells.

The case $m < n$ is symmetric.

Note that one letter repetition, e.g.

$$U(RU)_{n-1}RR(DR)_{m-1}D(LD)_{k-1}L(UL)_{l-1},$$

cannot happen: since the number of U and D, as well as the number of R and L, must be the same, we have $n+l=m+k$ and $n+m+1=k+l$, which is impossible since the sum of left-hand sides has different parity than the sum of right-hand sides.

Three letter repetitions, e.g.

$$U(RU)_{n-1}RR(DR)_{m-1}DD(LD)_{k-1}LL(UL)_{l-1},$$

are impossible due to similar arguments. Thus we have the following three cases, modulo rotation of the shape by 90° or 180°:

(3) Two letter repetitions, on top and bottom:

$$U(RU)_{n-1}RR(DR)_{m-1}D(LD)_{n-1}LL(UL)_{m-1}.$$

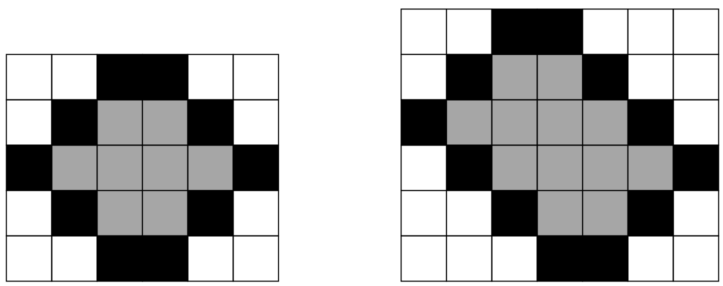


FIGURE 2. Case (2): $m=n$ and $m=n+1$, for $n=2$

If $m=n$, we have $4n+2$ Rock cells and $2+4+\cdots+2n+\cdots+4+2=2n^2$ Water cells.

If $m=n+1$, again, we can move all the Rock cells that are adjacent to the piece $D(LD)_{n-1}LL(LU)_{m-1}$ (the bottom half of the picture) and move them left; this reduces the case to Case (3).

If $m>n+1$, this configuration is not optimal since the same movement of Rock cells will add $m-n-1>0$ Water cells and will not change the number of Rock cells.

Similar arguments apply if $m<n$, and if we have repetitions UU, DD instead of RR, LL.

(3) Two letter repetitions on top and right of the shoreline:

$$U(RU)_{n-1}RR(DR)_{m-1}DD(LD)_{n-1}L(UL)_m.$$

If $m=n-1$, we have $4n+1$ Rock cells and $(2+\cdots+(2n-2))+((2n-1)+\cdots+1)=2n^2-n$ Water cells.

If $m=n$, we have $4n+3$ Rock cells and $(2+\cdots+2n)+((2n-1)+\cdots+1)=n(n+1)+n^2=2n^2+n$ Water cells.

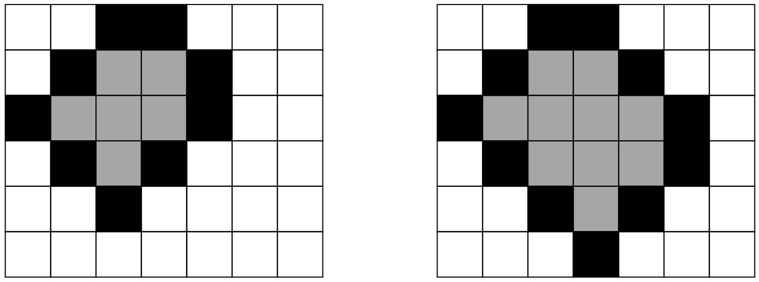


FIGURE 3. Case (3): $m = n - 1$ and $m = n$, for $n = 2$

If $m \geq n+1$, we can move the Rock cells adjacent to $D(LD)_{n-1}L(UL)_m$ to the left, increasing the number of Water cells by $m - n$. Similarly, if $m \leq n - 2$, we can move the Rock cells adjacent to $D(LD)_{n-1}L(UL)_mU$ to the right, increasing the number of Water cells by $n - m - 1$. So in these cases, the configuration is not optimal.

(4) Four repetitions RR, UU, DD, LL on all edges of the shoreline:

$$UU(RU)_{n-1}RR(DR)_{m-1}DD(LD)_{n-1}LL(UL)_{m-1}.$$

If $m = n$, we have $4n + 4$ Rock cells and $(2 + 4 + \cdots + 2n) + (2n + \cdots + 2) = 2n(n + 1)$ Water cells. Since we would get $(n+1)^2 + n^2 > 2n(n+1)$ Water cells with the diamond-shaped configuration of Case (1), this configuration is not optimal.

If $m = n + 1$, we have $4n + 6$ Rock cells and $2 + \cdots + 2n + (2n + 1) + 2n + \cdots + 2 = 2n^2 + 4n + 1$ Water cells. Since we would get $2(n + 1)^2 > 2n^2 + 4n + 1$ Water cells in case (2), this configuration is not optimal.

If $m - n \geq 2$, this shape is not optimal, the same motion of Rock cells as above will increase the number of Water cells by $m - n - 1$.

We conclude that:

- for $4n$ Rock cells, we have at most $n^2 + (n - 1)^2 = 2n^2 - 2n + 1$ Water cells;
- for $4n + 1$ Rock cells, we have at most $2n^2 - n$ Water cells;
- for $4n + 2$ Rock cells, we have at most $2n^2$ Water cells;
- for $4n + 3$ Rock cells, we have at most $2n^2 + n$ Water cells.

This implies the answer.

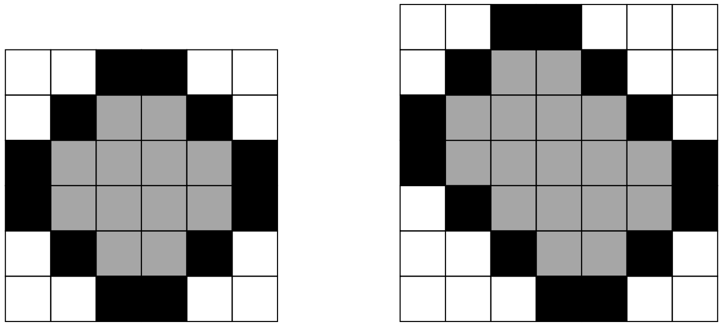


FIGURE 4. Case (4): $m = n$ and $m = n + 1$, for $n = 2$

(5) On Day 1, there are no Rock cells, and there are four Water cells forming a 2x2 square. On Day 1 and every day after that, after Water fills new cells, you are allowed to replace 3 Ground cells with Rock cells (note that you cannot put Rocks in Water). Show how you can stop the flood.

Solution.

			R6	R5	R4				
		R6	W6	W5	W4	R3			
	R6	W6	W5	W4	W3	W3	R3		
	R5	W5	W4	W3	W2	W2	R2		
R5	W5	W4	W3	W2	W1	W1	W2	R2	
	R4	W4	W3	W2	W1	W1	R1		
		R4	R3	R2	R1	R1			

FIGURE 5. Task 5: Stopping the flood in six days

(6) On Day 1, there is an infinite horizontal row of Rock cells. Right above them, there is a row of adjacent n Water cells. On Day 1 and on every subsequent day, after Water fills new cells, you are allowed to replace k Ground cells with Rock cells. Stop the flood for (a) $k = 3$; (b) $k = 2$.

(a) On Day 1, place two Rock cells adjacent to the leftmost water cell. Place one Rock cell to the right of the rightmost Water cell. On Day 2, Water will fill $n - 1$ cells on Row 2. Also, Row 1 of water is blocked.

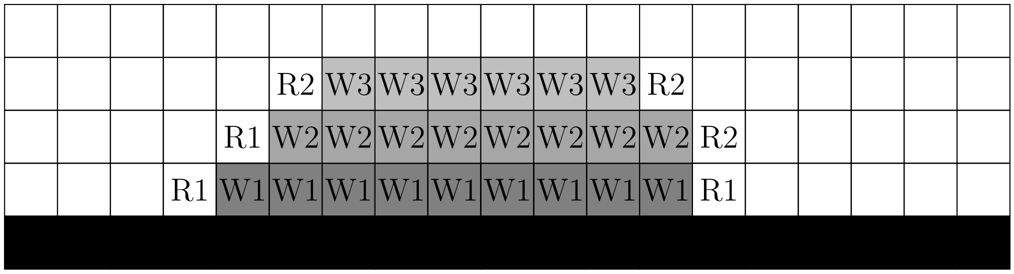


FIGURE 6. Stopping the flood in a half-plane, $k = 3$

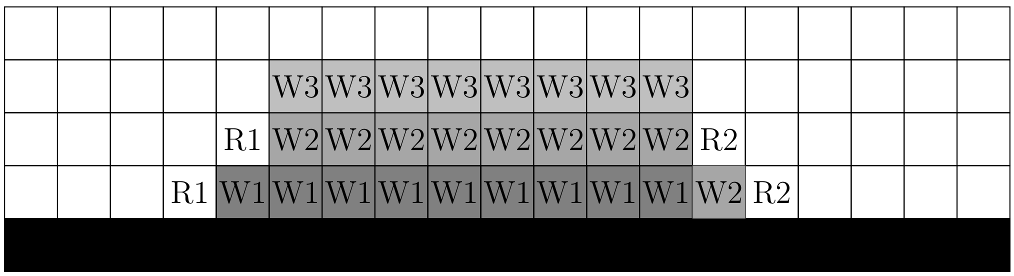


FIGURE 7. Stopping the flood in a half-plane, $k = 2$

On each of the next days, we will have a similar picture: a line of water of length $n - k$ on Row $k + 1$, blocked on one side. Each day, we will put 2 rock cells adjacent to the open end of the water line (third rock cell is not needed). This will block Row $k + 1$; next day, the water will fill $n - k - 1$ cells on Row $k + 2$, and this line will again be blocked on one side. Proceeding in this way, we will stop the flood in n days.

(b) On Day 1, place two Rock cells adjacent to the leftmost water cell. On Day 2, the Water will fill one cell in row 1 (near the right end) and $n - 1$ new cells in row 2. Place two Rock cells adjacent to the rightmost water cell. On Day 3, the water will fill $n - 1$ cells on Row 3.

Now, we have the same configuration in Row 3 as in the beginning, but with $n - 1$ cells instead of n. Rows 1 and 2 of Water are blocked. Proceeding in the same way for Row 3 instead of Row 1, we will stop the flood in $2n$ days.

(7) On Day 1, there are no Rock cells, and finitely many Water cells. Again, on Day 1 and on every subsequent day, after Water fills new cells, you are allowed to replace k Ground cells with Rock cells. For any initial configuration of Water cells, prove that you can stop the flood for (a) k=9; (b) k=5; (c*) k=4; (d*) k=3.

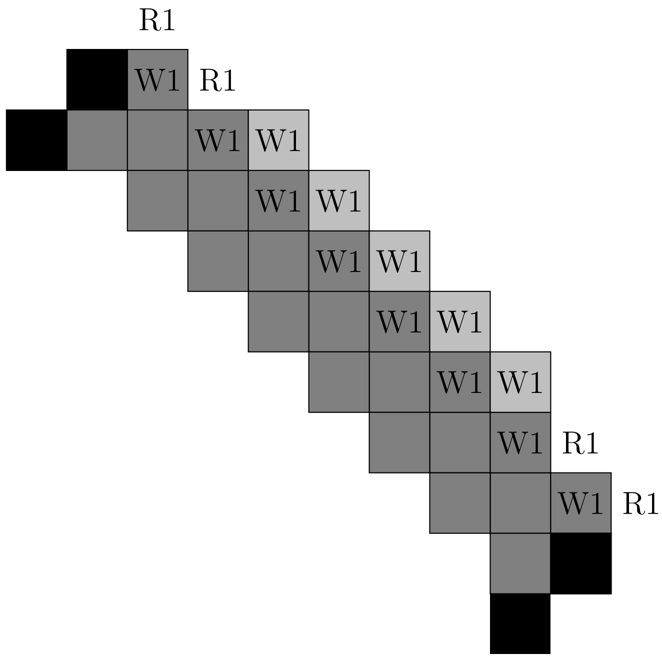


FIGURE 8. Stopping the flood, $k = 4$

(a) Suppose that initial configuration of Water fits in a $n \times n$ square. Then on Day N, the water will fit into a $(2N + n) \times (2N+n)$ square. We can surround it completely with $4(2N+n)$ Rock cells.

Select N so that $4(2N + n) < 9N$. Determine the shape of this square for that N, and start putting the rock cells to form a fence along its perimeter. We will complete this fence in at most N days, since we are placing 9 cells at a time and $4(2N + n) < 9N$. So when the water reaches the boundary of $(2N + n) \times (2N + n)$ square on Day N, the fence is ready and the flood will stop once the water fills the interior of this square.

(b) Suppose that initial configuration of Water fits in the same diamond shape as is Task 1. We will call it a diamond D_n if it has $2n + 1$ cells on its longest row. Then on Day N, the water will fit into D_{N+n}. We can surround it completely with $2(2(N + n) + 1) + 2$ Rock cells (these are 2 rock cells per row, plus 2 on top and bottom).

Select N so that $2(2(N+n)+1)+2 < 5N$, and build the fence in the same way as in (a) along D_{N+n}. In N days, the fence will be ready and the Water will be confined to the diamond D_{N+n}.

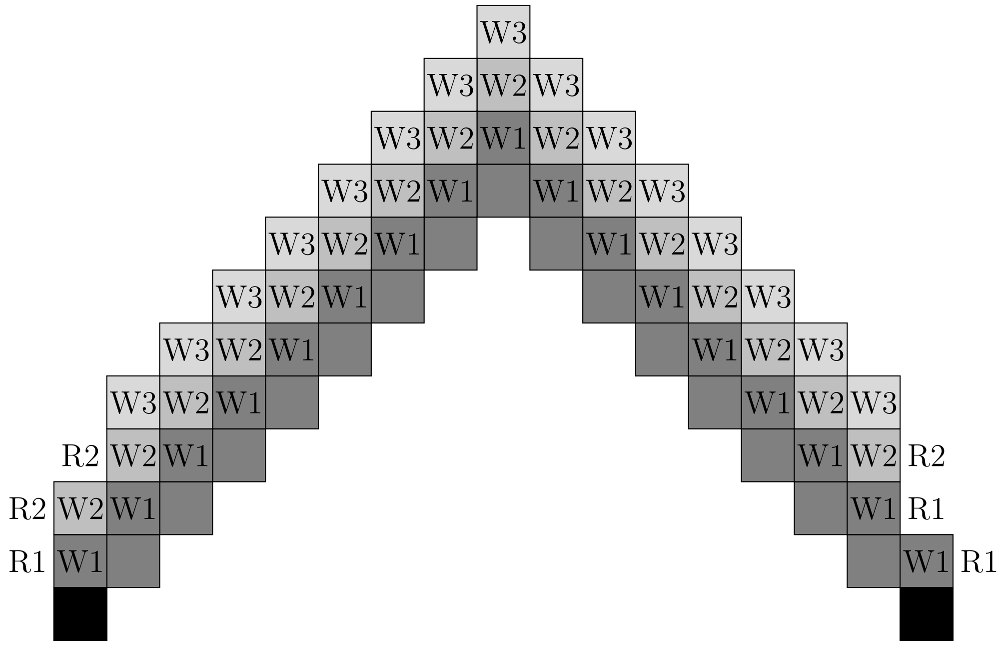


FIGURE 9. Stopping the flood, $k = 3$

(c) Let D_n be the diamond shape that contains the initial water configuration. Select N such that $3(N+n)+3 < 4N$ and build the fence along three sides of D_{N+n}.

On Day N, the water will reach the fence, and three of its sides will be complete since they contain $3(N+n)+3$ cells. The water will continue to spill through the fourth side.

On each of the subsequent days, Water cells that are adjacent to Ground will form a ladder. By placing two cells on the bottom of the ladder and two on top (see the picture), we reduce the length of this ladder by 2 on each step. Thus the flood will stop in finitely many days.

(d)* Let D_n be the diamond shape that contains the initial water configuration. We will assume that the initial water configuration is exactly D_n (if it is smaller, the same algorithm works but we have less Water). Select N such that $2(N+n)+3 < 2N$ and build the fence along two bottom sides of D_{N+n}.

When the water reaches the fence, two of its bottom sides are ready. In the next two days, we put 6 cells near the left and the right of the diamond as shown. Now, we are in the same situation as before, with half-a-diamond shape of water still spilling; but the size of the half-a-diamond shape is 1 smaller than before. Thus we can proceed in the same way until we stop the flood.