Solutions to Best Student Exam
Texas A&M High School Math Contest
October 25, 2025

1. Find the minimal value of $\sqrt{x^2+(1-y)^2}+\sqrt{(1-x)^2+y^2}$, where x, y are real numbers.

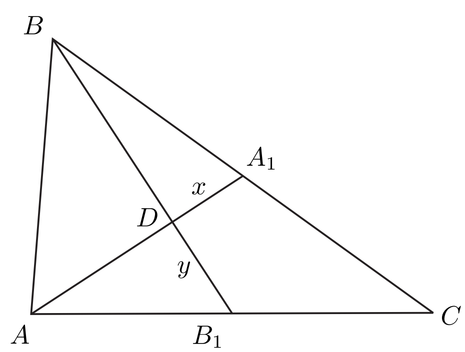
The distance from (x, y) to $(0, 1)$ is equal to $\sqrt{x^2+(1-y)^2}$. The distance from (x, y) to $(1, 0)$ is equal to $\sqrt{(1-x)^2+y^2}$. Therefore, the minimal value of this sum is the distance from $(1, 0)$ to $(0, 1)$.
Answer: $\sqrt{2}$.

2. In $\triangle ABC$ the median from A is perpendicular to the median from B. Find AB if $BC = 7$ and $AC = 6$.

Let AA_1 and BB_1 be the medians, and let D be the point of their intersection. Denote $DA_1 = x$ and $DB_1 = y$. Then $AD = 2x$ and $BD = 2y$. Pythagorean theorem for $\triangle BDA_1$ and $\triangle ADB_1$ gives us

$$\begin{cases} x^2 + 4y^2 = \frac{7^2}{2^2} \\ 4x^2 + y^2 = 3^2 \end{cases}$$

Which gives $x^2 + y^2 = \frac{1}{5}\left(\frac{49}{4}+9\right) = \frac{17}{4}$. Pythagorean theorem for $\triangle ABD$ gives $AB^2 = 4x^2 + 4y^2 = 17$.



Answer: $\sqrt{17}$.

3. Two particles move along the edges of equilateral $\triangle ABC$ in the direction $A \to B \to C \to A$ starting simultaneously and moving at the same speed. One starts at A, and the other starts at the midpoint of $\overline{BC}$. The midpoint of the line segment joining the two particles traces out a path that encloses a region R. What is the ratio of the area of R to the area of $\triangle ABC$?

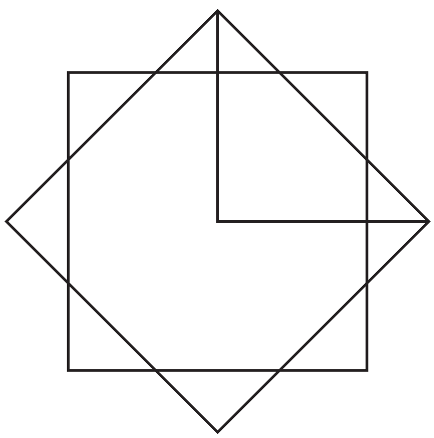
It is clear that the midpoint traces a straight line (i.e., has constant velocity) when both particles trace straight lines (have constant velocity). If one particle is in a vertex of the triangle, then the other particle is in the middle of the opposite side. It follows that the region R is the triangle with the vertices equal to the midpoints of the medians of $\triangle ABC$. The point of intersection of the medians divides them in proportion $1 : 2$. Consequently,

the distance from the point of intersection of the median to the vertices of the triangle R is $1/2 - 1/3 = 1/6$ of the median. The distance from the intersection of the medians to a vertex of $\triangle ABC$ is $2/3$ of the median. It follows that R is obtained from $\triangle ABC$ by similarity with the coefficient $\frac{1/6}{2/3} = \frac{1}{4}$. Hence the ratio of the area of R to the area of $\triangle ABC$ is $\frac{1}{16}$.
Answer: $1/16$.

4. A standard 8×8 chess board is rotated around its center by $45°$. Find the area of the set of the points of the plane which belonged to black squares in both the original and the rotation positions of the board, if the side of the board length $1 ft$.

Let us denote by XY, where X and Y are letters B or W (standing for "black" and "white", respectively), the area of the set of points which are in a square of color X in the original board and color Y in the rotated board.

If we rotate one of the boards by $90°$, then the colors of the squares will change to the opposite ones. It follows that $BB = WB$, $BW = WW$, $WB = WW$, and $BB = BW$. Hence $BB = WB = WW = BW$. Consequently, BB is equal to one quarter of the area of intersection of two boards.



The part of a board that does not belong to the intersection consists of isosceles right triangles with height $\frac{\sqrt{2}}{2} - \frac{1}{2}$. Hence, the area of each of these triangles is $\left(\frac{\sqrt{2}-1}{2}\right)^2 = \frac{3-2\sqrt{2}}{4}$.

Consequently, the area of the intersection is $1 - (3 - 2\sqrt{2}) = 2\sqrt{2} - 2$. It follows that the answer is $\frac{\sqrt{2}-1}{2}$.
Answer: $\frac{\sqrt{2}-1}{2}$ ft^2.

5. Let $\lfloor x \rfloor$ denote the largest integer not larger than x, and let $\{x\} = x - \lfloor x \rfloor$. Find the number of real solutions of the equation

$$\lfloor x \rfloor \cdot \{x\} + x = 2\{x\} + 10.$$

Denote $\lfloor x \rfloor = n$ and $\{x\} = \alpha$. We have

$$n\alpha + n + \alpha = 2\alpha + 10,$$

which is equivalent to $(n-1)\alpha = 10 - n$. Since $0 \le \alpha$, the equality implies that $n > 1$. Hence, $(n-1)\alpha$ is non-negative, so $n \le 10$.

Since $0 \le \alpha < 1$, we have $n = \dfrac{\alpha + 10}{1+\alpha} = 1 + \dfrac{9}{1+\alpha} > 1 + \frac{9}{2}$, so $n \ge 6$.

Checking $n \in \{6, 7, 8, 9, 10\}$, we see that the solutions are $x = n + \frac{10-n}{n-1}$ for $n \in \{6, 7, 8, 9, 10\}$.

Answer: 5.

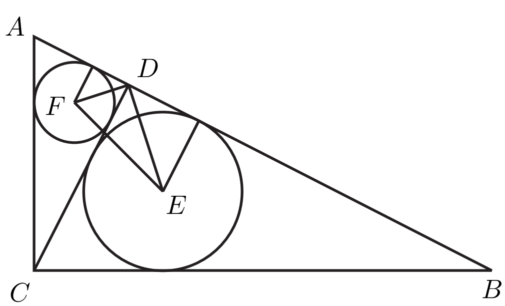
6. Find all two-digit decimal numbers $\overline{xy}$ such that their square is equal to the cube of the sum of its digits (i.e., to $(x+y)^3$).

The equality $(10x+y)^2 = (x+y)^3$ implies that $x+y$ is a full square. It also implies that $x+y$ and $(x+y)^2$ give the same residue when divided by 3. Consequently, the residue of $x+y$ modulo 3 is either 0 or 1. It follows that $x+y$ is one of the numbers $4, 9, 16$. We have $10x+y = (x+y)^{3/2}$, and $4^{3/2} = 8$, $9^{3/2} = 27$, $16^{3/2} = 64$. Out of these three numbers, only 27 satisfies the condition of the problem.

Answer: 27.

7. Let $\triangle ABC$ be such that the angle $\angle ACB$ is right, and the radius of the inscribed circle of $\triangle ABC$ is 1. Let CD be the height of $\triangle ABC$. Suppose that the bisector of $\angle ABC$ intersects the bisector of $\angle BCD$ in E, and that the bisector of $\angle BAC$ intersects the bisector of $\angle ACD$ in F. Find EF.

The points E and F are the centers of the inscribed circles in $\triangle BCD$ and $\triangle ACD$, respectively. Suppose that r_1, r_2 are their radii. Since DE and DF are bisectors of $\angle CDB$ and $\angle ADC$, respectively, $\angle EDF$ is right. The segments ED and EF have lengths $r_1\sqrt{2}$ and $r_2\sqrt{2}$. Consequently, $EF = \sqrt{2(r_1^2 + r_2^2)}$.

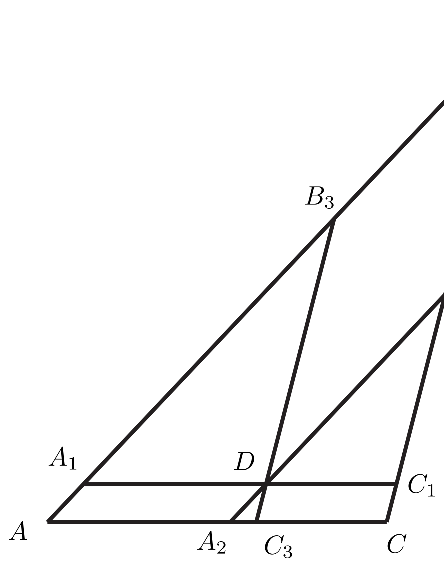


Since $\triangle ABC$, $\triangle ACD$, and $\triangle CBD$ are similar, there exists k such that their areas are kr^2, kr_1^2, and kr_2^2, respectively, where r is the radius of the inscribed circle in $\triangle ABC$. It follows that $r^2 = r_1^2 + r_2^2$. Consequently, $EF = \sqrt{2}r = \sqrt{2}$.

Answer: $\sqrt{2}$

8. Three lines ℓ_1, ℓ_2, ℓ_3 are parallel to the sides of $\triangle ABC$ and intersect in one point in its interior. All three segments of ℓ_i formed by the two intersection points of ℓ_i with the sides of $\triangle ABC$ have the same length x. Find x if the lengths of the sides of $\triangle ABC$ are $2, 3, 4$.

Let D be the intersection point of the lines ℓ_i, and let $A_1, A_2, B_2, B_3, C_1, C_3$ be the intersections of the lines with the sides of the triangle so that $\ell_1 = \overleftrightarrow{A_1C_1}$, $\ell_2 = \overleftrightarrow{A_2B_2}$, $\ell_3 = \overleftrightarrow{B_3C_3}$, as it is shown on the figure.



Let $a = BC, b = AC, c = AB$. Since BB_2DB_3, CC_3DC_1, and AA_2DA_1 are parallelograms, their opposite sides are congruent, hence $C_1B_2 = a - x$, $A_2C_3 = b - x$, and $A_1B_3 = c - x$. Since $\triangle ABC$ and $\triangle A_2B_2C$ are similar,

$$\frac{CB_2}{CB} = \frac{A_2B_2}{AB},$$

i.e.,

$$CB_2 = \frac{CB \cdot A_2B_2}{AB} = \frac{ax}{c}.$$

Consequently, $BB_2 = a - \frac{ax}{c}$. Similarly, $CC_1 = a - \frac{ax}{b}$. We have $a = BC = BB_2 + B_2C_1 + C_1C = a - \frac{ax}{c} + a - x + a - \frac{ax}{b}$, hence

$$\frac{ax}{c} + \frac{ax}{b} + x = 2a,$$

i.e.,

$$\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 2.$$

Consequently, $x = \frac{2}{1/a+1/b+1/c} = \frac{2}{1/2+1/3+1/4} = \frac{24}{13}$.

Answer: $\frac{24}{13} = 1\frac{11}{13}$.

9. For a positive integer n, let $S(n)$ denote the sum of decimal digits of n. Find the number of solutions of the equation $n + S(n) + S(S(n)) = 2025$.

The residues of $S(n)$ and n modulo 3 are equal, the same is true for $S(n)$ and $S(S(n))$. Since 2025 is divisible by 9, this implies that n is divisible by 3. We have $S(n) \leq 1+9+9+9 = 28$, so $S(S(n)) \leq 10$. Consequently, $n \geq 2025 - 28 - 10 = 1987$. We also have $n < 2025$. Numbers divisible by 3 in this interval, the corresponding values of $S(n)$ and $S(S(n))$, and the sum $n + S(n) + S(S(n))$ are

1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019	2022
27	21	24	27	3	6	9	3	6	9	12	6
9	3	6	9	3	6	9	3	6	9	3	6
2025	2016	2025	2034	2007	2016	2025	2016	2025	2034	2034	2034

Answer: 4.

10. How many numbers α exist such that $0 \le \alpha < 2\pi$ and all numbers

$$\cos\alpha, \cos 2\alpha, \cos 4\alpha, \ldots, \cos 2^n\alpha, \ldots$$

are negative?

Let us write the number $\frac{\alpha}{2\pi}$ in the binary numeration system $.a_1a_2\ldots$. The set of values of $\frac{\alpha}{2\pi}$ for which $\cos\alpha$ is negative is equal to the interval $(1/4, 3/4)$, which is the set of numbers which can not be of the form $.00a_3\ldots$ or $.11a_3\ldots$. Since the binary expansion of $\frac{2^n\alpha}{2\pi}$ is $.a_{n+1}a_{n+2}\ldots$, it follows that the set of angles α such that $\cos 2^n\alpha < 0$ for all n is equal to the set of angles α such that in the binary expansion $.a_1a_2\ldots$ of $\frac{\alpha}{2\pi}$ there are no two consecutive equal digits. Consequently, there are only two such numbers, corresponding to the binary fractions $.010101\ldots$ and $.101010\ldots$, which are $\alpha = \frac{2\pi}{3}$ and $\alpha = \frac{4\pi}{3}$.
Answer: 2.

11. Evaluate $\int_0^\pi (|\sin 2025x| - |\sin 2026x|)\, dx$.

Since $\sin(\alpha+\pi) = -\sin\alpha$, the function $|\sin\alpha|$ is periodic with period π. Consequently, $|\sin kx|$ is periodic with period π/k, hence $\int_0^\pi |\sin kx|\, dx = k\int_0^{\pi/k} \sin kx\, dx = \int_0^\pi \sin y\, dy = 2$, where $y = x/k$.
Answer: 0.

12. Denote by a_n the integer closest to $\sqrt{n}$. Find $\displaystyle\sum_{n=1}^{2025}\frac{1}{a_n}$.

We have $a_n = k$ if and only if $k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2}$ (the equalities can not hold, since $\sqrt{n}$ is either an integer or is irrational). These inequalities are equivalent to

$$k^2 - k + \frac{1}{4} < n < k^2 + k + \frac{1}{4}$$

and to

$$k^2 - k + 1 \le n \le k^2 + k.$$

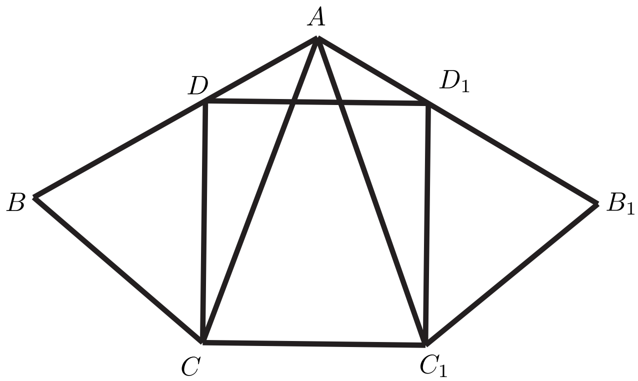
Denote $b_k = k^2 + k = k(k+1)$. Then the set of values of n such that $a_n = k$ is the interval $[b_{k-1}+1, b_k] \cap \mathbb{N}$.

The sum of the values of $\frac{1}{a_n}$ on the interval $[b_{k-1}+1, b_k]$ is equal to $\frac{b_k - b_{k-1}}{k} = \frac{k(k+1)-k(k-1)}{k} = 2$. We have $a_{2025} = 45$, and $2025 \in [b_{44}+1, b_{45}] = [1981, 2070]$.

It follows that $\displaystyle\sum_{n=1}^{2025}\frac{1}{a_n} = 2 \times 44 + \frac{2025-1981+1}{45} = 88 + \frac{45}{45} = 89$.
Answer: 89.

13. In $\triangle ABC$, we have $\angle A = 40°, \angle B = \angle C$. Point D on the side AB is such that $\frac{BC}{AD} = \sqrt{3}$. Find $\angle DCB$ in degrees.

Let us construct triangles $\triangle ACC_1$ and $\triangle AC_1B_1$ congruent to $\triangle ABC$, as it is shown on the figure.



Let D_1 be a point on AB_1 such that $\overleftrightarrow{DD_1}$ is parallel to $\overleftrightarrow{CC_1}$. Then $\angle BAB_1 = 120°$, and $\angle ADD_1 = \angle AD_1D = 30°$. Consequently, $\frac{DD_1}{AD} = \sqrt{3}$, which implies that $DD_1 = BC = CC_1$. Since the perpendicular from A to the segments $\overline{DD_1}$ and to $\overline{CC_1}$ intersect them in their midpoints, this implies that DD_1C_1C is a rectangle. It follows that $\angle DCB = 140° - 90° = 50°$.
Answer: $50°$.

14. Find all odd non-prime numbers n such that $(n-1)!$ is not divisible by n^2.

If $(n-1)!$ is not divisible by n^2, then there exists a prime p dividing n such that if p^k is the largest power of p dividing n, then the largest power of p dividing $(n-1)!$ is less than $2k$. Since n is odd and not prime, we have either $p^k \le n/3$ or $p^k = n$. In the first case, $p^k, 2p^k \le n-1$, so $(n-1)!$ is divisible by p^{2k}, which is a contradiction. In the second case, the maximal power of p dividing $(p^k-1)!$ is $\left\lfloor \frac{p^k-1}{p} \right\rfloor + \left\lfloor \frac{p^k-1}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{p^k-1}{p^{k-1}} \right\rfloor = (p^{k-1}-1)+(p^{k-2}-1)+\cdots+(p-1)$.

We have $k \ge 2$. If $p > 3$, i.e., $p \ge 5$, then the above sum is not less than $4+24(k-2) \ge 2k$. It follows that $p = 3$. If $k \ge 3$, then the sum is not less than $2+8+26(k-3) \ge 2k$. Hence, the only possibility is $n = 3^2 = 9$. It is easy to see that $8!$ is not divisible by 9^2.
Answer: 9.

15. Let $y = ax + b$ be the line tangent to the graph of $y = x(x-1)(x-2)(x-4)$ in two points. Find a.

If the line $y = ax+b$ is tangent to the graph in two points, then $x(x-1)(x-2)(x-4)-ax-b$ is a square of a quadratic polynomial (since it will have two double roots). Therefore, there exist real numbers p, q such that

$$x(x-1)(x-2)(x-4) - ax - b = (x^2 + px + q)^2,$$

hence

$$x^4 - 7x^3 + 14x^2 - (8+a)x - b = x^4 + 2px^3 + (p^2 + 2q)x^2 + 2pqx + q^2.$$

Which takes place if and only if

$$2p = -7, \quad p^2 + 2q = 14, \quad 8 + a = -2pq, \quad b = -q^2.$$

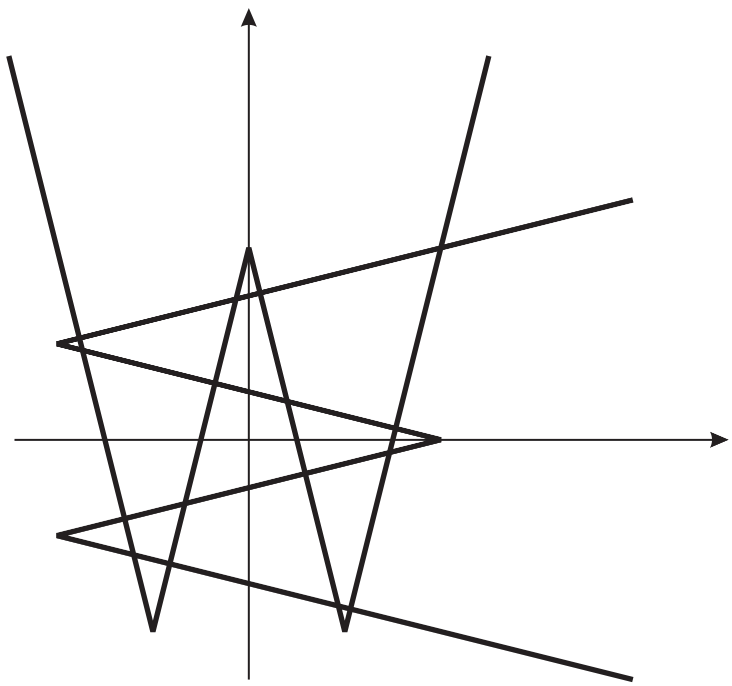
The first two equalities imply that $p = -\frac{7}{2}$, and $q = \frac{1}{2}\left(14 - \frac{49}{4}\right) = \frac{7}{8}$. The third equation gives:

$$a = -2pq - 8 = \frac{49}{8} - 8 = -\frac{15}{8}.$$

Answer: $-\frac{15}{8} = -1\frac{7}{8} = -1.875$.

16. Consider the function $f(x) = |4 - 4|x|| - 2$. Find the number of solutions of the equation $f(f(x)) = x$.

A number x is a solution of the equation $f(f(x)) = x$ if and only if there exists y such that both (x, y) and (y, x) belong to the graph of the function $f(x)$. It follows that the number of solutions is equal to the number of intersections of the graphs of $y = |4 - 4|x|| - 2$ and $x = |4 - 4|y|| - 2$.



Answer: 16.

17. Find x from the system

$$\frac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} = 1, \qquad \frac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} = \frac{1}{\sqrt{2}}.$$

We have

$$\frac{(x - y\sqrt{x^2 - y^2})^2}{1 - x^2 + y^2} = 1, \qquad \frac{(y - x\sqrt{x^2 - y^2})^2}{1 - x^2 + y^2} = \frac{1}{2}$$

$$\frac{x^2 - 2xy\sqrt{x^2 - y^2} + x^2y^2 - y^4}{1 - x^2 + y^2} = 1, \quad \frac{y^2 - 2xy\sqrt{x^2 - y^2} + x^4 - x^2y^2}{1 - x^2 + y^2} = \frac{1}{2}.$$

Subtracting the second equation from the first, we get

$$\frac{1}{2} = \frac{x^2 - y^2 - x^4 + 2x^2y^2 - y^4}{1 - x^2 + y^2} = \frac{x^2 - y^2 - (x^2 - y^2)^2}{1 - x^2 + y^2} = \frac{(x^2 - y^2)(1 - (x^2 - y^2))}{1 - x^2 + y^2} = x^2 - y^2.$$

Substituting this result into the original equations, we get

$$\frac{x - y \cdot \frac{1}{\sqrt{2}}}{1/\sqrt{2}} = 1, \qquad \frac{y - x \cdot \frac{1}{\sqrt{2}}}{1/\sqrt{2}} = \frac{1}{\sqrt{2}},$$

i.e., the system

$$\left\{ \begin{array}{rcrcl} x\sqrt{2} & - & y & = & 1 \\ -x & + & y\sqrt{2} & = & 1/\sqrt{2}. \end{array} \right.$$

Multiplying the first equation by $\sqrt{2}$, and adding the result to the second equation, we get

$$x = \sqrt{2} + \frac{1}{\sqrt{2}}.$$

Answer: $\sqrt{2} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$.

18. Positive real numbers x, y satisfy $x^2 + y^2 + \frac{1}{x^2+2x} + \frac{1}{y^2+2y} = 2$. Find $x + y$.

Let us show that $x^2 + \frac{1}{x^2+2x} \geq 1$ for all positive x. It is enough to show that $x^2(x^2 + 2x) + 1 \geq x^2 + 2x$ for all x. But this follows from the equality

$$x^4 + 2x^3 - x^2 - 2x + 1 = (x^2 + x - 1)^2.$$

It also follows that we have $x^2 + \frac{1}{x^2+2x} = 1$ for a positive x if and only if x is a positive root of $x^2 + x - 1$, i.e., if and only if $x = \frac{\sqrt{5}-1}{2}$.

Consequently, $x^2 + y^2 + \frac{1}{x^2+2x} + \frac{1}{y^2+2y} = 2$ for positive x and y if and only if $x = y = \frac{\sqrt{5}-1}{2}$.
Answer: $x + y = \sqrt{5} - 1$.

19. Let $a < b < c$ be the roots of $x^3 - 3x + 1 = 0$. Find $a^2 - c$.

If a is a root of $x^3 - 3x + 1$, then we have

$$(a^2 - 2)^3 - 3(a^2 - 2) + 1 = a^6 - 6a^4 + 9a^2 - 1 = (a^3 - 3a + 1)^2 - 2(a^3 - 3a - 1) = 0.$$

It follows that if a is a root of the polynomial $x^3 - 3x + 1$, then $a^2 - 2$ is also its root, i.e., $a^2 - 2 \in \{a, b, c\}$. We can not have $a^2 - 2 = a$, since the roots of $a^2 - a - 2$ are $\frac{1 \pm 3}{2}$, and are not roots of $x^3 - 3x + 1$.

The local extrema of the function $x^3 - 3x + 1$ are the roots of $3x^2 - 3$, i.e., -1 and 1. It follows that the function is increasing on $(-\infty, -1]$, decreasing on $[-1, 1]$, and increasing on $[1, \infty)$. In particular, $a \in (-\infty, -1)$, $b \in (-1, 1)$, and $c \in (1, \infty)$.

We have $(-\sqrt{3})^3 - 3(-\sqrt{3}) + 1 = 1 > 0$, hence $a \in (-\infty, -\sqrt{3})$. The function $x^2 - 2$ is decreasing on $(-\infty, 0)$. It follows that $a^2 - 2 > (-\sqrt{3})^2 - 2 = 1$. Therefore, $a^2 - 2 = c$. It follows that $a^2 - c = 2$.

Answer: 2.

20. A cube with side of length 5 is partitioned into 5^3 little cubes with sides of length 1. We choose k little cubes, and draw three lines through the center of each chosen cube parallel to its edges. What is the smallest number k for which we can choose the little cubes in such a way that the lines intersect all 5^3 little cubes?

Consider the bottom 5×5 square of the cube. Let us write in each of its 25 little squares the number of chosen cubes above it. Suppose that some square, call it S, has 0 written in it. Since each of the cubes above S must be intersected by a line passing through a chosen cube, the sum of numbers written in the "cross" formed by the row and the column containing S must be at least 5.

Consider the 10 sums of numbers written in rows and columns of the square, and choose the smallest number m out of them. Without loss of generality, let us assume that m is the sum of numbers in the first row of the square. Suppose that $m < 5$. Then the first row has at least $5 - m$ zeros, and the sum of numbers in each column containing one of these zeros is at least $5 - m$, by the proven in the previous paragraph. The sum of numbers in each of the other columns is at least m. It follows that the sum of all numbers in the square is at least $(5-m)^2 + m^2 \geq 5^2/2$, hence $(5-m)^2 + m^2 \geq 13$. If $m \geq 5$, then the total sum is at least 25.

Consequently, the number of chosen cubes must be at least 13. We can choose 13 cubes, for example, in the following way:

		4	3	5
		3	5	3
		5	4	3
1	2			
2	1			

where the numbers show in which layer we choose a cube above the corresponding square. Empty cells do not have chosen cubes above them.

Answer: 13.