Solutions to CD Exam

Texas A&M High School Math Contest

25 October, 2025

1. In a triangle ABC with $AB = 5, BC = 6$, and $AC = 7$, points D and E lie on $\overline{AC}$ with $AD = 1$ and $EC = 2$. Find the area of $\triangle BDE$.

   Answer: $\dfrac{24\sqrt{6}}{7}$

   Solution: Let s be the semiperimeter and $[\triangle ABC]$ be the area of $\triangle ABC$.

$$s = \frac{AB + BC + CA}{2} = \frac{5 + 6 + 7}{2} = 9.$$

   By Heron's formula,

$$[\triangle ABC] = \sqrt{9(9-5)(9-6)(9-7)} = \sqrt{216} = 6\sqrt{6}.$$

   Thus,

$$\frac{[\triangle BDE]}{[\triangle BAC]} = \frac{DE}{AC} = \frac{4}{7}, \quad \text{or} \quad [\triangle BDE] = 6\sqrt{6} \cdot \frac{4}{7} = \frac{24\sqrt{6}}{7}.$$

2. Solve for x: $\quad 12x^{7/5} + 3x^{2/5} = 13x^{9/10}$.

   Answer: $0, \dfrac{1}{9}, \dfrac{9}{16}$

   Solution:

$$\begin{aligned} 12x^{7/5} + 3x^{2/5} &= 13x^{9/10} \\ 12x^{14/10} + 3x^{4/10} &= 13x^{9/10} \\ x^{4/10}(12x - 13x^{1/2} + 3) &= 0. \end{aligned}$$

   At this point we can conclude that one of the solutions is $x = 0$ based on the $x^{4/10}$ factor. For the portion in parentheses, we will use the substitution $u = x^{1/2}$ and rewrite as follows:

$$\begin{gathered} 12u^2 - 13u + 3 = 0 \\ (4u - 3)(3u - 1) = 0 \\ u = \tfrac{3}{4}, \quad u = \tfrac{1}{3}. \end{gathered}$$

   Substituting $x^{1/2}$ back in for u and solving for x, we see that $x = u^2$. Therefore,

$$x = \left(\tfrac{3}{4}\right)^2, \quad x = \left(\tfrac{1}{3}\right)^2$$

   The complete solution set is

$$\{0, \tfrac{9}{16}, \tfrac{1}{9}\}.$$

3. Find the largest solution x of the equation $\big|3|x| - 2\big| = 1 - 2x$.

   Answer: $-1/5$

   Solution: First consider the case $x \geq 0$. In this case $|x| = x$ and the equation is simplified to $|3x - 2| = 1 - 2x$. Since $3x - 2 = 0$ for $x = 2/3$, we need to consider two subcases: $0 \leq x < 2/3$ and $x \geq 2/3$.

   If $x \geq 2/3$ then the equation is further simplified to $3x - 2 = 1 - 2x$, which has solution $x = 3/5$. However, we have to drop this solution (at least in this subcase) since it does not belong to the interval $[2/3, \infty)$.

   If $0 \leq x < 2/3$ then the equation becomes $-(3x - 2) = 1 - 2x$, which has solution $x = 1$. This one has to be dropped too as it does not belong to the interval $[0, 2/3)$.

   Now consider the case $x < 0$. In this case $|x| = -x$ and the equation is simplified to $|-3x - 2| = 1 - 2x$, which is equivalent to $|3x + 2| = 1 - 2x$. Again, there are two subcases to consider: $-2/3 \leq x < 0$ and $x < -2/3$.

   If $-2/3 \leq x < 0$ then the equation is further simplified to $3x + 2 = 1 - 2x$, which has solution $x = -1/5$. This time $-1/5$ does belong to the interval of interest $[-2/3, 0)$. Hence it is also a solution of the original equation.

   Finally, if $x < -2/3$, any solution will be smaller than $-1/5$. Therefore, the largest solution is $-1/5$.

4. Given $b_1 > b_2$, let $f_1(x) = 20x^2 + b_1x + 250$ and $f_2(x) = 5x^2 + b_2x - 125$. Given that the parabolas intersect in exactly one point find the value of $b_1 - b_2$.

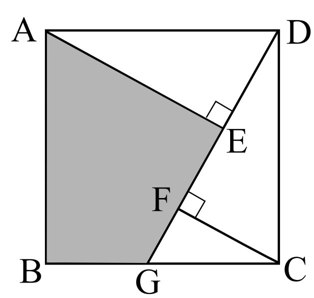
   Answer: 150

   Solution: Let x_0 denote the x-coordinate of the point of intersection of f_1 and f_2. Since f_1 and f_2 have exactly one common point we have

$$f_1(x) - f_2(x) = 15x^2 + (b_1 - b_2)x + 375 = 15(x - x_0)^2 = 15x^2 - 30xx_0 + 15x_0^2.$$

   It follows that $b_1 - b_2 = -30xx_0$ and $375 = 15x_0^2 \Rightarrow x_0 = \pm 5$ Since $b_1 - b_2 > 0$, $x_0 = -5$ and $b_1 - b_2 = 150$.

5. The figure below shows a square $ABCD$. Find the area of the shaded region if $EF = 3$ and $DG = 13$.



Answer: 51.

Solution: Since $\triangle ADE$ is congruent to $\triangle DCF$, $DE = FC$ and $AE = DF$. Let $DE = x$ then $FG = 10 - x$. From the similarity $\triangle DFC \sim \triangle CFG$, we have

$$\frac{FC}{DF} = \frac{FG}{FC} \quad \Rightarrow \quad \frac{x}{3+x} = \frac{10-x}{x} \quad \Rightarrow \quad 2x^2 - 7x - 30 = (x-6)(2x+5) = 0 \quad \Rightarrow \quad x = 6$$

We have $AE = DF = 6 + 3 = 9$ and

$$DC = AD = \sqrt{6^2 + 9^2} = 3\sqrt{13}, \quad GC = \frac{2}{3}DC = 2\sqrt{13}$$

The area of shaded region is

$$(3\sqrt{13})^2 - \frac{AE \cdot ED}{2} - \frac{GC \cdot CD}{2} = 117 - 27 - \frac{2\sqrt{13} \cdot 3\sqrt{13}}{2} = 51.$$

6. Find a real solution x of the equation $\sqrt{x\sqrt{x\sqrt{x}}} = 8\sqrt{2}$.

Answer: 16.

Clearly, any real solution should be nonnegative. For any $x \geq 0$ we obtain

$$\begin{aligned}
\sqrt{x} &= x^{1/2}, \\
x\sqrt{x} &= x \cdot x^{1/2} = x^{3/2}, \\
\sqrt{x\sqrt{x}} &= \left(x^{3/2}\right)^{1/2} = x^{3/4}, \\
x\sqrt{x\sqrt{x}} &= x \cdot x^{3/4} = x^{7/4}, \\
\sqrt{x\sqrt{x\sqrt{x}}} &= \left(x^{7/4}\right)^{1/2} = x^{7/8}.
\end{aligned}$$

Hence the given equation is equivalent to $x^{7/8} = 8\sqrt{2}$. We observe that

$$8\sqrt{2} = 2^3 \cdot 2^{1/2} = 2^{7/2} = \left(2^4\right)^{7/8} = 16^{7/8}.$$

Since the function $f(x) = x^{7/8}$ is strictly increasing on the interval $[0, \infty)$, it follows that $x = 16$ is the only real solution of the equation.

7. Among the 9 points with integer coordinates (x, y) with $0 \leq x \leq 2$ and $0 \leq y \leq 2$, how many distinct pentagons can be formed by choosing 5 of these points as vertices such that all interior angles are strictly less than $180°$?

Answer: 20

Solution: Label the 3×3 grid points. The center point $(1, 1)$ cannot be a vertex of a pentagon with the condition. Hence the vertices must be chosen from the 8 outer points.

A 5-point subset of these 8 outer points forms a convex pentagon if and only if no three of the chosen points are collinear. Among the outer 8 points, the only collinear triples are the four side-triples of the bounding square:

$$\{(0,0),(1,0),(2,0)\}, \quad \{(0,2),(1,2),(2,2)\}, \quad \{(0,0),(0,1),(0,2)\}, \quad \{(2,0),(2,1),(2,2)\}.$$

There are $\binom{8}{5} = 56$ ways to choose 5 of the outer 8 points in total. Let $A_1, \ldots, A_4$ be the events that our 5-set contains all three points of the bottom, top, left, and right side, respectively. Then $|A_i| = \binom{5}{2} = 10$ for each i (after fixing one side's triple, choose the remaining 2 from the other 5 points), so $\sum |A_i| = 4 \cdot 10 = 40$.

We consider intersections of events. Two opposite sides cannot both be fully included in a 5-set, but two *adjacent* sides can; in that case the 5 points are forced (the union of the two side-triples has size 5), so $|A_i \cap A_j| = 1$ for each of the 4 adjacent pairs, giving $\sum_{i<j} |A_i \cap A_j| = 4$. Any intersection of 3 or 4 sides is impossible in a 5-set.

By inclusion–exclusion, the number of valid 5-subsets is

$$\binom{8}{5} - \sum |A_i| + \sum_{i<j} |A_i \cap A_j| = 56 - 40 + 4 = 20.$$

8. Let $a > b > 0$, $a^2 + b^2 = 3ab$. Find the value of $\dfrac{a+b}{a-b}$. Write your answer as a single term with no sums or differences..

Answer: $\sqrt{5}$

Solution: From $a^2 + b^2 = 3ab$, we get $\left(\frac{a}{b}\right)^2 - 3\left(\frac{a}{b}\right) + 1 = 0$. Solving the equation for $\dfrac{a}{b}$, we have $\dfrac{a}{b} = \dfrac{3 \pm \sqrt{5}}{2}$. Since $a > b > 0$, $\dfrac{a}{b}$ must be greater than 1, so $\dfrac{a}{b} = \dfrac{3+\sqrt{5}}{2}$. Therefore,

$$\frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \frac{5+\sqrt{5}}{1+\sqrt{5}}$$

Rationalizing the denominator gives us $\dfrac{5 - 4\sqrt{5} - 5}{1-5} = \sqrt{5}$.

9. A polynomial $f(x) = x^5 - 2x^4 + ax^2 + bx$, where a and b are unknown coefficients, is divisible by $x^2 - 3x + 2$. Find ab.

Answer: -2.

The polynomial $x^2 - 3x + 2$ has roots 1 and 2. Since the polynomial $f(x)$ is divisible by it, we have $f(x) = (x^2 - 3x + 2)q(x)$ for some polynomial $q(x)$. It follows that 1 and 2 are also roots of $f(x)$. The equalities $f(1) = f(2) = 0$ give rise to a system of linear equations in variables a and b:

$$\begin{cases} -1 + a + b = 0 \\ 4a + 2b = 0 \end{cases} \iff \begin{cases} a + b = 1 \\ b = -2a \end{cases} \iff \begin{cases} -a = 1 \\ b = -2a \end{cases} \iff \begin{cases} a = -1 \\ b = 2 \end{cases}$$

Thus $ab = -1 \cdot 2 = -2$.

10. If $i = \sqrt{-1}$, find the sum

$$1 + i + i^2 + i^3 + \cdots + i^{2025}$$

Answer: $1 + i$

Solution: For each nonnegative integer k, we have that $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, and $i^{4k+3} = -i$. This implies that $i^{4k} + i^{4k+1} + i^{4k+2} + i^{4k+3} = 0$, so the sum is equal to $i^{2024} + i^{2025} = 1 + i$.

11. Find all solutions x of the equation $\log_3 x + \log_x 27 = \log_3(9x^3) + \log_x(9/x)$.

Answer: $\frac{1}{3}$; $\sqrt{3}$.

Let us introduce a new variable $y = \log_3 x$. Then $3^y = x$. Based on the second and fourth logarithm in the equation, $x \neq 1$ so $y \neq 0$. Therefore, we have $x^{1/y} = 3$ so that $1/y = \log_x 3$. We obtain that

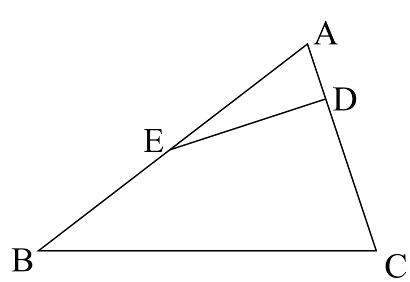
$$\begin{aligned}
&\log_x 27 = \log_x(3^3) = 3/y, \\
&\log_3(9x^3) = 2\log_3 3 + 3\log_3 x = 2 + 3y, \\
&\log_x(9/x) = \log_x(3^2) + \log_x(x^{-1}) = 2/y - 1.
\end{aligned}$$

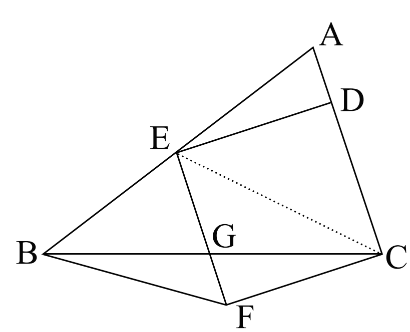
Now the equation can be rewritten as $y + 3/y = 3y + 2/y + 1$. After simplification, $2y^2 + y - 1 = 0$. This quadratic equation has two solutions, $y = -1$ and $y = 1/2$. Since both solutions are different from 0, it follows that $x = 3^{-1}$ and $x = 3^{1/2}$ are solutions of the original equation.

12. In a triangle ABC, points D and E lie on sides $\overline{AC}$ and $\overline{AB}$ respectively. It is given that $\angle ADE = 90°$ and $\angle AED = 30°$. If $\overline{BE} \cong \overline{ED} \cong \overline{DC}$, find the measure of $\angle ACB$.

Answer: $75°$

Solution: Let F be the point so that $\overline{EF} \parallel \overline{AC}$ and $\overline{FC} \parallel \overline{ED}$ as in the figure below.





A parallelogram $DEFC$ has an interior angle $\angle EDC = 90°$ and congruent sides $\overline{ED} \cong \overline{DC}$. Observe that $\square DEFC$ is a square. Since $\triangle BFE$ is an equilateral triangle,

$$\angle BFC = \angle BFE + \angle EFC = 60° + 90° = 150°.$$

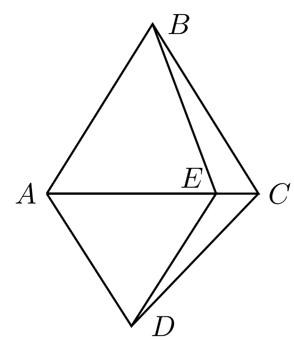
On the isosceles triangle $\triangle BFC$, we have

$$\angle BCF = \frac{180° - 150°}{2} = 15°.$$

Therefore,

$$\angle ACB = 90° - \angle BCF = 75°.$$

13. For two equilateral triangles $\triangle ABC$ and $\triangle ADE$, find $\angle BEC$ if $\angle CDE = 14°$.



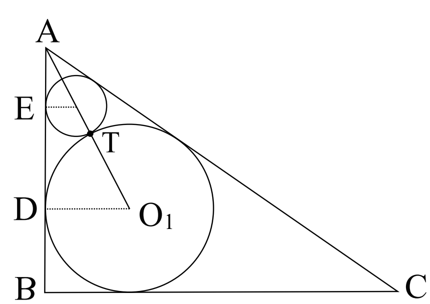
Answer: 106°

Solution: Since $AB = AC$, $AE = AD$, and $\angle BAE \cong \angle CAD$, we have congruent triangles $\triangle BAE \cong \triangle CAD$. ($\triangle CAD$ is the rotation of $\triangle BAE$ by 60° clockwise) This implies $\angle AEB \cong \angle ADC = 60° + 14° = 74°$, or $\angle BEC = 180° - 74° = 106°$.

14. In a right triangle ABC with $\angle A = 60°$, the inscribed circle C_1 has radius 1. A circle C_2 is tangent to $\overline{AB}$, $\overline{AC}$, and externally tangent to C_1. Find the radius of C_2.

Answer: $\dfrac{1}{3}$.

Solution: Let O_1 and O_2 be the centers of C_1 and C_2, and D and E be the tangent points of C_1 and C_2 with $\overline{AB}$ as in the figure below.



Since $\angle DAO_1 = 30°$, $AO_1 = 2$ and $AD = \sqrt{3}$. For the tangent point T of C_1 and C_2, we have

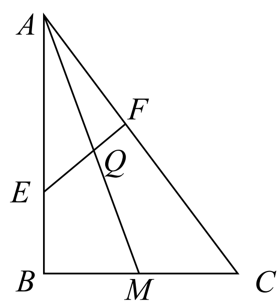
$$AT = AO_1 - O_1T = 2 - 1 = 1.$$

Let r be the radius of C_2. From the similarity $\triangle AO_1D \sim \triangle AO_2E$, we have

$$\frac{AO_1}{AO_2} = \frac{AD}{AE} \quad \Leftrightarrow \quad \frac{2}{2-1-r} = \frac{\sqrt{3}}{r\sqrt{3}} \quad \Leftrightarrow \quad 2r\sqrt{3} = (1-r)\sqrt{3},$$

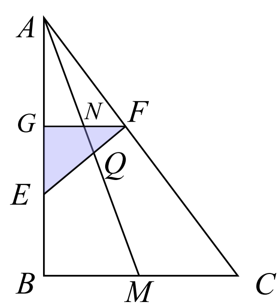
which implies $r = \dfrac{1}{3}$.

15. The triangle ABC has sides $BC = 3$, $BA = 4$, and $AC = 5$. Consider points E and F on $\overline{\text{AB}}$ and $\overline{\text{AC}}$ with $AE : AF = 3 : 2$. Find the ratio $\dfrac{QE}{QF}$ if $BM = MC$ and Q is the intersection of $\overline{\text{EF}}$ and $\overline{\text{AM}}$.



Answer: $\dfrac{15}{8}$

Solution: Let G be a point on $\overline{AB}$ such that $\overline{GF}$ is parallel to $\overline{BC}$ and let N be the intersection of $\overline{GF}$ and $\overline{AM}$ as in the figure below.



Applying the Theorem of Menelaus to $\triangle GEF$ and a transversal line $\overline{AM}$ (based on a point K on $\overline{EF}$ with $\overline{GK} \parallel \overline{AQ}$), we have

$$\frac{GA}{AE} \cdot \frac{EQ}{QF} \cdot \frac{FN}{NG} = 1 \tag{1}$$

Since two triangles $\triangle AGF$ and $\triangle ABC$ are similar, we have $AG : GF : FA = 4 : 3 : 5$ and so

$$\frac{GA}{AE} = \frac{GA}{AF} \cdot \frac{AF}{AE} = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$$

Moreover, by the similarity again, we have

$$\frac{FN}{NG} = \frac{CM}{MB} = \frac{1.5}{1.5} = 1.$$

Thus the identity (1) above implies

$$\frac{GA}{AE} \cdot \frac{EQ}{QF} \cdot \frac{FN}{NG} = 1 \quad \Rightarrow \quad \frac{8}{15} \cdot \frac{EQ}{QF} \cdot 1 = 1 \quad \Rightarrow \quad \frac{EQ}{QF} = \frac{15}{8}$$

16. Find all solutions of the equation

$$\sqrt{x + \sqrt{4x-3} + \frac{1}{4}} + \sqrt{x - \sqrt{4x-3} + \frac{1}{4}} = x.$$

Answer: 3

Solution: The equation is equivalent to $\sqrt{\left(\frac{\sqrt{4x-3}}{2} + 1\right)^2} + \sqrt{\left(\frac{\sqrt{4x-3}}{2} - 1\right)^2} = x$. So we get

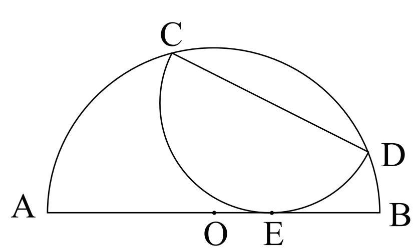
$$\left|\frac{\sqrt{4x-3}}{2} + 1\right| + \left|\frac{\sqrt{4x-3}}{2} - 1\right| = x.$$

Note that the first absolute value term is positive whenever defined ($x \geq \frac{3}{4}$). For the second we have

$$\left|\frac{\sqrt{4x-3}}{2} - 1\right| = \begin{cases} \frac{\sqrt{4x-3}}{2} - 1, & \text{if } \frac{3}{4} \leq x < \frac{7}{4} \\ -\left(\frac{\sqrt{4x-3}}{2} - 1\right), & \text{if } x \geq \frac{7}{4} \end{cases}$$

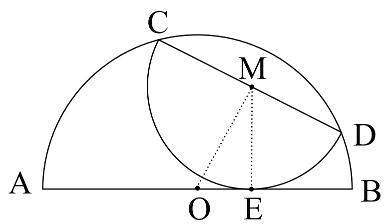
Thus, if $\frac{3}{4} \leq x < \frac{7}{4}$ the equation becomes $\frac{\sqrt{4x-3}}{2} + 1 - \frac{\sqrt{4x-3}}{2} + 1 = x$, so $x = 2$, which is impossible. If $\frac{7}{4} \leq x$, then the equation becomes $\frac{\sqrt{4x-3}}{2} + 1 + \frac{\sqrt{4x-3}}{2} - 1 = x$, so we need to solve $\sqrt{4x-3} = x$. Squaring both sides we get $4x - 3 = x^2$, with solutions 3 and 1. Since $x \geq \frac{7}{4}$, only $x = 3$ is a solution.

17. Points C and D lie on the arc of a semicircle with diameter $\overline{AB}$. Let O be the midpoint of $\overline{AB}$. The circle with diameter $\overline{CD}$ is tangent to the segment $\overline{AB}$ at a point E as in the figure below. Given $CD = 12$ and $OE = 1$, find the value of AB^2.



Answer: 292

Solution: Let the semicircle S_1 have center $O = (0,0)$ and radius R. Without loss of generality, we may assume the the configuration of two semi circles as in the figure below since the tangency point E has coordinate either $E = (-1,0)$ or $E = (0,1)$ and it doesn't affect the AB.



Let M be the midpoint of $\overline{CD}$. The semicircle S_2 with diameter $\overline{CD}$ has the center M and radius $r = \frac{CD}{2} = 6$. Tangency to the line $\overline{AB}$ (the x-axis) implies that the perpendicular distance from M to $\overline{AB}$ equals r; hence $ME = 6$.

On $\triangle MOE$, we have

$$OM^2 = 1^2 + 6^2 = 37.$$

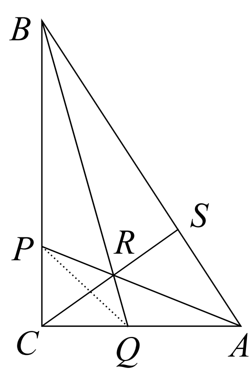
Moreover, on the cord $\overline{CD}$ of S_1 (or on the right triangle $\triangle OMC$ or $\triangle OMD$), we have

$$R^2 = OM^2 + OC^2 = 37 + OA^2 = 37 + 6^2 = 73.$$

Therefore, $AB = 2R$ and

$$AB^2 = (2R)^2 = 4R^2 = 4 \cdot 73 = 292.$$

18. Let $\triangle ABC$ be the triangle with $AB = 10$, $AC = 6$ and $\angle C = 90$. Take two points P and Q on sides of $\overline{\text{BC}}$ and $\overline{\text{CA}}$ with $CP = CQ = 2$. Two lines $\overline{\text{AP}}$ and $\overline{\text{BQ}}$ intersect on R and the line $\overline{\text{CR}}$ and $\overline{\text{AB}}$ meet on S. Find TS if T is the intersection of $\overline{\text{PQ}}$ and $\overline{\text{BA}}$.



Answer: 24

Solution: By the Pythgorean theorem, we have $BC = 8$. To find AS, apply the theorem of Ceva to $\triangle BCA$ and three lines $\overline{BQ}, \overline{CS}$, and $\overline{AP}$;

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AS}{SB} = 1 \quad \Rightarrow \quad \frac{6}{2} \cdot \frac{2}{4} \cdot \frac{AS}{SB} = 1 \quad \Rightarrow \quad \frac{AS}{SB} = \frac{2}{3}$$

Since $AB = 10$, we have $AS = 4$.

The point T is on the extension of the line $\overline{BA}$. To find AT, apply the theorem of Menelaus to $\triangle BCA$ and the line $\overline{PQT}$. From the identity

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AT}{TB} = 1 \quad \Rightarrow \quad \frac{6}{2} \cdot \frac{2}{4} \cdot \frac{AT}{TB} = 1 \quad \Rightarrow \quad \frac{AT}{TB} = \frac{2}{3},$$

we have

$$\frac{AT}{TA + AB} = \frac{AT}{TA + 10} = \frac{2}{3} \quad \Rightarrow \quad AT = 20.$$

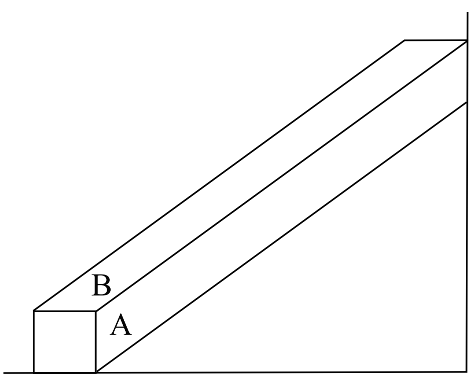
Thus $TS = TA + AS = 20 + 4 = 24$.

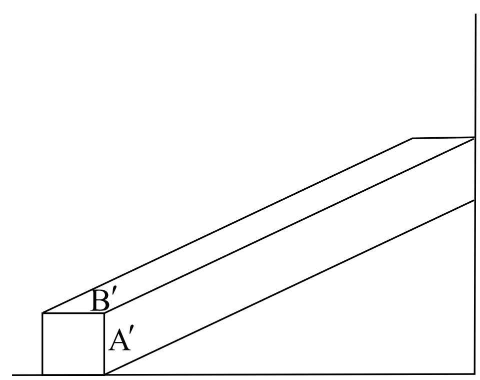
19. A cube moves along a straight line perpendicular to the front face, so that it collects rain only on its top and front faces—not on the lateral faces. The cube travels 12 meters with an initial speed of 1 m/s, while rain falls vertically at 6 m/s and is evenly distributed. If the cube's speed is increased by 50% to reduce the amount of rain collected, what is the percentage decrease in the total rain collected? Round your answer to the nearest tenth of a percent.

Answer: 28.6%

Solution: The amount of rain collected is determined by the volume of space swept by the cube's top and front faces. Since raindrops are evenly distributed, comparing these volumes is sufficient.

Consider the rain that hits the front face during the journey. The raindrops are contained within the space modeled as parallelogram A shown in the figure on the left.





This parallelogram A has an area (representing volume) of $s \cdot 12$, where s is the side of the cube. Similarly, the rain that hit the top face are in the space corresponding to the parallelogram B. The area of B is

$$s \cdot 12 \cdot 6 = 72s.$$

The right figure illustrates parallelograms A' and B' that contain rain hitting the front and top faces, respectively, with the increased speed of 1.5 m/s.

The area of A' is the same as the area of A while B' has the area

$$s \cdot 8 \cdot 6 = 48s.$$

Now we compare the amount of rain the cube collects at the new speed $R_{1.5}$ to the initial amount R_1:

$$R_{1.5} : R_1 = (12s + 48s) : (12s + 72s) = \frac{5}{7}$$

The increased speed reduces the amount of rain collected by 28.6%, since

$$\frac{2}{7} = 0.2857\cdots$$

20. From a point A outside a circle C, two tangents are drawn to the circle, touching it at points P and Q. Through P, draw a line parallel to $\overline{AQ}$, which meets the circle again at R. Let the line $\overline{AR}$ intersect the circle again at S. If $AP : PR = 2 : 3$ and the area of $\triangle ASQ$ is 50, find the area of quadrilateral $APRQ$.

    Answer: 500

    Solution: Figure 1 (a) illustrates the quadrilateral $APRQ$. As in Figure 1 (b), we draw secant lines $\overline{PS}$ and $\overline{SQ}$ to have

$$\angle PRA \cong \angle APS := x \text{ and } \angle QRA \cong \angle AQS := y$$

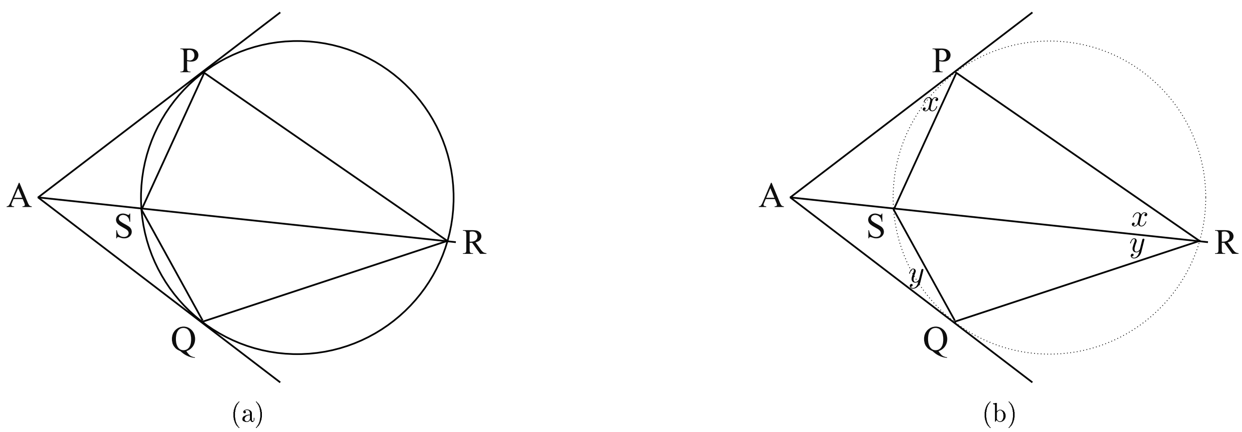


Figure 1: Initial stage with tangent and secant lines.

We need to examine angles further. A transversal $\overline{QR}$ passing parallel lines $\overline{PR}$ and $\overline{AQ}$ implies

$$\angle AQR = 180^\circ - \angle PRQ = 180^\circ - (x + y),$$

and so the sum of interior angles of $\triangle AQR$ yields

$$\angle QAR = 180^\circ - (\angle AQR + \angle QRA) = 180^\circ - (180^\circ - (x + y)) - y = x.$$

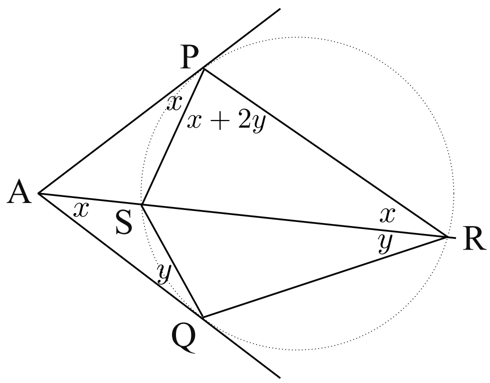
The sum of angels in the inscribed quadrilateral $\square PSQR$ is

$$\begin{aligned}&\angle RPS + \angle PSQ + \angle SQR + \angle QRP \\ =&\angle RPS + (180^\circ - (x+y)) + (180^\circ - (x+y) - y) + (x+y) = 360^\circ,\end{aligned}$$

which implies

$$\angle RPS = x + 2y. \tag{2}$$

Figure 2 (a) below shows angles in x and y that we have found so far.



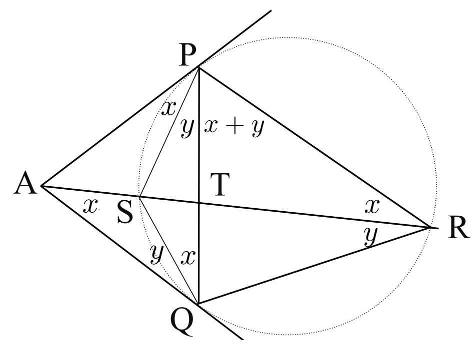


Figure 2: Further congruent angles

As in Figure 2 (b) above, draw a secant line $\overline{PQ}$. By Tangent–Secant Angle Theorem applied to the tangents $\overline{AP}$ and $\overline{AQ}$, and a secant line $\overline{PQ}$, we have

$$\angle APQ \cong \angle AQP \cong \angle PRQ = x + y.$$

Consequently,

$$\angle SPQ = \angle APQ - \angle APS = (x+y) - x = y \tag{3}$$

and similarly,

$$\angle SQP = \angle AQP - \angle AQS = (x+y) - y = x$$

Now (2) and (3) imply

$$\angle RPQ = \angle RPS - \angle SPQ = x + y.$$

We have similar triangles

$$\triangle ASQ \sim \triangle QSP \text{ and } \triangle QPR \sim \triangle AQP \text{ (isosceles)} \tag{4}$$

together with

$$\triangle APR \sim \triangle ASP \tag{5}$$

The similarity (5) and the condition that $AP : PR = 2 : 3$ imply

$$\frac{AP}{PR} = \frac{2}{3} = \frac{AS}{PS}.$$

The first similarity of (4) implies

$$\frac{AS}{QS} = \frac{QS}{PS} \quad \Rightarrow \quad QS^2 = AS \cdot PS \quad \Rightarrow \quad \left(\frac{QS}{AS}\right)^2 = \frac{PS}{AS} = \frac{3}{2} \tag{6}$$

The area of $\triangle QSP$, $[\triangle QSP]$, satisfies

$$\frac{[\triangle QSP]}{[\triangle ASQ]} = \left(\frac{QS}{AS}\right)^2 \quad \Rightarrow \quad [\triangle QSP] = \frac{3}{2}50 = 75.$$

Transversal lines $\overline{AR}$ and $\overline{PQ}$ passing parallel lines $\overline{PR}$ and $\overline{AQ}$ determine a similar triangles $\triangle TPR \sim \triangle TQA$. Two tangent lines $\overline{AP}$ and $\overline{AQ}$ are congruent and so the similarity ratio implies

$$\frac{QT}{PT} = \frac{AQ}{PR} = \frac{2}{3} \quad \Rightarrow \quad \frac{[\triangle PST]}{[\triangle QST]} = \frac{AQ}{PR} = \frac{2}{3}.$$

Now we have the area of $\triangle AQT$:

$$[\triangle AQT] = [\triangle ASQ] + [\triangle QST] = 50 + \frac{2}{5}[\triangle QSP] = 50 + \frac{2}{5}75 = 80.$$

Similarly, the area of $\triangle PAT$ is

$$[\triangle AQT] = \frac{3}{2}[\triangle AQT] = 120.$$

Next, we find the area of $\triangle QPR$, which is determined by the second similarity ratio in (4). By the same argument as in (6), we have

$$\frac{AP}{PQ} = \frac{PQ}{PR} \quad \Rightarrow \quad \frac{PQ^2}{AP^2} = \frac{3}{2}$$

Therefore, the area of $\triangle QPR$ is

$$[\triangle QPR] = \frac{3}{2}[\triangle APQ] = \frac{3}{2}(80 + 120) = 300.$$

The area of quadrilateral $APRQ$ is

$$[\triangle APQ] + [\triangle QPR] = 200 + 300 = 500.$$