Solutions to DE Exam

Texas A&M High School Math Contest

25 October, 2025

1. A rectangle has perimeter 17 and area 15. Find the length of the diagonal of this rectangle.

   Answer: $\dfrac{13}{2} = 6.5$.

   Let a and b be lengths of two adjacent sides of the rectangle. Then the rectangle has perimeter $2(a+b)$ and area ab, and its diagonals have length $\sqrt{a^2+b^2}$. We are given that $2(a+b) = 17$ and $ab = 15$. It follows that $b = 15/a$ and $2(a+15/a) = 17$. Likewise, $2(b+15/b) = 17$. Hence a and b are solutions of the equation $2(x+15/x) = 17$. This equation is equivalent to a quadratic equation $2x^2 - 17x + 30 = 0$, which has solutions $5/2$ and 6. Note that $(5/2)^2 \neq 15$ and $6^2 \neq 15$. Hence one of the lengths a and b is $5/2$ and the other is 6. Then the diagonals of the rectangle have length $\sqrt{(5/2)^2+6^2} = \sqrt{169/4} = 13/2$.

2. Solve for x: $\quad 12x^{7/5} + 3x^{2/5} = 13x^{9/10}$.

   Answer: $0, \dfrac{1}{9}, \dfrac{9}{16}$

   Solution:

$$12x^{7/5} + 3x^{2/5} = 13x^{9/10}$$
$$12x^{14/10} + 3x^{4/10} = 13x^{9/10}$$
$$x^{4/10}(12x - 13x^{1/2} + 3) = 0.$$

   At this point we can conclude that one of the solutions is $x = 0$ based on the $x^{4/10}$ factor. For the portion in parentheses, we will use the substitution $u = x^{1/2}$ and rewrite as follows:

$$12u^2 - 13u + 3 = 0$$
$$(4u-3)(3u-1) = 0$$
$$u = \tfrac{3}{4}, \quad u = \tfrac{1}{3}.$$

   Substituting $x^{1/2}$ back in for u and solving for x, we see that $x = u^2$. Therefore,

$$x = \left(\tfrac{3}{4}\right)^2, \quad x = \left(\tfrac{1}{3}\right)^2$$

   The complete solution set is

$$\{0, \tfrac{9}{16}, \tfrac{1}{9}\}.$$

3. Find the largest solution x of the equation $\big|3|x|-2\big| = 1-2x$.

   Answer: $-1/5$

   Solution: First consider the case $x \geq 0$. In this case $|x| = x$ and the equation is simplified to $|3x-2| = 1-2x$. Since $3x-2=0$ for $x = 2/3$, we need to consider two subcases: $0 \leq x < 2/3$ and $x \geq 2/3$.

   If $x \geq 2/3$ then the equation is further simplified to $3x-2 = 1-2x$, which has solution $x = 3/5$. However, we have to drop this solution (at least in this subcase) since it does not belong to the interval $[2/3, \infty)$.

   If $0 \leq x < 2/3$ then the equation becomes $-(3x-2) = 1-2x$, which has solution $x = 1$. This one has to be dropped too as it does not belong to the interval $[0, 2/3)$.

   Now consider the case $x < 0$. In this case $|x| = -x$ and the equation is simplified to $|-3x-2| = 1-2x$, which is equivalent to $|3x+2| = 1-2x$. Again, there are two subcases to consider: $-2/3 \leq x < 0$ and $x < -2/3$.

   If $-2/3 \leq x < 0$ then the equation is further simplified to $3x+2 = 1-2x$, which has solution $x = -1/5$. This time $-1/5$ does belong to the interval of interest $[-2/3, 0)$. Hence it is also a solution of the original equation.

   Finally, if $x < -2/3$, any solution will be smaller than $-1/5$. Therefore, the largest solution is $-1/5$.

4. Given $b_1 > b_2$, let $f_1(x) = 20x^2 + b_1x + 250$ and $f_2(x) = 5x^2 + b_2x - 125$. Given that the parabolas intersect in exactly one point find the value of $b_1 - b_2$.

   Answer: 150

   Solution: Let x_0 denote the x-coordinate of the point of intersection of f_1 and f_2. Since f_1 and f_2 have exactly one common point we have

   $$f_1(x) - f_2(x) = 15x^2 + (b_1 - b_2)x + 375 = 15(x - x_0)^2 = 15x^2 - 30xx_0 + 15x_0^2.$$

   It follows that $b_1 - b_2 = -30xx_0$ and $375 = 15x_0^2 \Rightarrow x_0 = \pm 5$ Since $b_1 - b_2 > 0$, $x_0 = -5$ and $b_1 - b_2 = 150$.

5. What is the constant term when the expression $\left(x + 1 + \dfrac{1}{x}\right)^4$ is expanded?.

   Answer: 19.

   The expanded expression is the sum of $3^4 = 81$ products of the form $y_1y_2y_3y_4$, where each of the factors y_i can be chosen to be either x or 1 or $\dfrac{1}{x}$. Every product is equal to x^k for some integer k. Hence c_0 counts the number of choices when the product $y_1y_2y_3y_4$ equals 1. We have three different ways to arrange this. First we can choose all four factors to be 1. Secondly, we can choose one of the four factors to be x, one of the remaining three factors to be $\dfrac{1}{x}$, and the other two factors to be 1. There are $4 \cdot 3 = 12$ different choices here. Finally, we can choose two of the four factors to be x and the other two factors to be $\frac{1}{x}$. There are $\binom{4}{2} = \frac{4 \cdot 3}{2!} = 6$ different choices here. Thus $c_0 = 1 + 12 + 6 = 19$.

6. Find a real solution x of the equation $\sqrt{x\sqrt{x\sqrt{x}}} = 8\sqrt{2}$.

   Answer: 16.

   Clearly, any real solution should be nonnegative. For any $x \geq 0$ we obtain

$$\sqrt{x} = x^{1/2},$$
$$x\sqrt{x} = x \cdot x^{1/2} = x^{3/2},$$
$$\sqrt{x\sqrt{x}} = \left(x^{3/2}\right)^{1/2} = x^{3/4},$$
$$x\sqrt{x\sqrt{x}} = x \cdot x^{3/4} = x^{7/4},$$
$$\sqrt{x\sqrt{x\sqrt{x}}} = \left(x^{7/4}\right)^{1/2} = x^{7/8}.$$

   Hence the given equation is equivalent to $x^{7/8} = 8\sqrt{2}$. We observe that

$$8\sqrt{2} = 2^3 \cdot 2^{1/2} = 2^{7/2} = \left(2^4\right)^{7/8} = 16^{7/8}.$$

   Since the function $f(x) = x^{7/8}$ is strictly increasing on the interval $[0, \infty)$, it follows that $x = 16$ is the only real solution of the equation.

7. Evaluate the sum $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{2025}{1012^2 \cdot 1013^2}$.

   Answer: $1 - \frac{1}{1013^2} = \frac{1026168}{1026169}$

   Solution: The sum has 1012 terms. The nth term is

$$\frac{2n+1}{n^2(n+1)^2}.$$

   For any n we have

$$\frac{2n+1}{n^2(n+1)^2} = \frac{(n^2+2n+1) - n^2}{n^2(n+1)^2} = \frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

   Therefore the sum equals

$$\left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{1012^2} - \frac{1}{1013^2}\right) = 1 - \frac{1}{1013^2} = \frac{1026168}{1026169}.$$

   Note that $1013^2 = (1000+13)^2 = 1000^2 + 2 \cdot 13 \cdot 1000 + 13^2 = 10^6 + 26 \cdot 10^3 + 169 = 1\,026\,169$.

8. Let $a > b > 0$, $a^2 + b^2 = 3ab$. Find the value of $\dfrac{a+b}{a-b}$. Write your answer as a single term with no sums or differences..

Answer: $\sqrt{5}$

Solution: From $a^2 + b^2 = 3ab$, we get $\left(\frac{a}{b}\right)^2 - 3\left(\frac{a}{b}\right) + 1 = 0$. Solving the equation for $\dfrac{a}{b}$, we have $\dfrac{a}{b} = \dfrac{3 \pm \sqrt{5}}{2}$. Since $a > b > 0$, $\dfrac{a}{b}$ must be greater than 1, so $\dfrac{a}{b} = \dfrac{3+\sqrt{5}}{2}$. Therefore,

$$\frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \frac{5+\sqrt{5}}{1+\sqrt{5}}$$

Rationalizing the denominator gives us $\dfrac{5 - 4\sqrt{5} - 5}{1-5} = \sqrt{5}$.

9. A polynomial $f(x) = x^5 - 2x^4 + ax^2 + bx$, where a and b are unknown coefficients, is divisible by $x^2 - 3x + 2$. Find ab.

Answer: -2.

The polynomial $x^2 - 3x + 2$ has roots 1 and 2. Since the polynomial $f(x)$ is divisible by it, we have $f(x) = (x^2 - 3x + 2)q(x)$ for some polynomial $q(x)$. It follows that 1 and 2 are also roots of $f(x)$. The equalities $f(1) = f(2) = 0$ give rise to a system of linear equations in variables a and b:

$$\begin{cases} -1 + a + b = 0 \\ 4a + 2b = 0 \end{cases} \iff \begin{cases} a + b = 1 \\ b = -2a \end{cases} \iff \begin{cases} -a = 1 \\ b = -2a \end{cases} \iff \begin{cases} a = -1 \\ b = 2 \end{cases}$$

Thus $ab = -1 \cdot 2 = -2$.

10. If $i = \sqrt{-1}$, find the sum

$$1 + i + i^2 + i^3 + \cdots + i^{2025}$$

Answer: $1 + i$

Solution: For each nonnegative integer k, we have that $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, and $i^{4k+3} = -i$. This implies that $i^{4k} + i^{4k+1} + i^{4k+2} + i^{4k+3} = 0$, so the sum is equal to $i^{2024} + i^{2025} = 1 + i$.

11. Determine how many real solutions (x, y) the following system of equations has:

$$\begin{cases} x^2 + y^2 = 18, \\ \sin(x - y) = 0. \end{cases}$$

Answer: 6.

The equation $\sin z = 0$ has solutions $z = \pi n$, where n is an arbitrary integer. Therefore $x - y = \pi n$, $n \in \mathbb{Z}$. Note that

$$(x+y)^2 + (x-y)^2 = (x^2 + 2xy + y^2) + (x^2 - 2xy + y^2) = 2(x^2 + y^2).$$

Hence the condition $x^2 + y^2 = 18$ implies that $(x-y)^2 \le 36$. Since $(2\pi)^2 > 6^2 = 36$, it follows that $x - y = 0$ or π or $-\pi$. In the case $x - y = 0$, we obtain two solution of the system: $x = y = 3$ and $x = y = -3$.

In the case $x-y = \pi$, let us substitute $y+\pi$ for x into the first equation of the system: $(y+\pi)^2+y^2 = 18$. After simplification, $2y^2 + 2\pi y + (\pi^2 - 18) = 0$. Since $\pi^2 - 18 < 0$, it is clear that the quadratic equation has two real solutions y_1 and y_2. This yields two solutions of the system: $(y_1 + \pi, y_1)$ and $(y_2 + \pi, y_2)$. As for the case $x - y = -\pi$, we notice that whenever (x_0, y_0) is a solution of the system, so is (y_0, x_0). It follows that the only solutions of the system in this case are $(y_1, \pi + y_1)$ and $(y_2, \pi + y_2)$. Thus the system has a total of six real solutions.

12. Find all solutions x of the equation $\log_3 x + \log_x 27 = \log_3(9x^3) + \log_x(9/x)$.

Answer: $\frac{1}{3}$; $\sqrt{3}$.

Let us introduce a new variable $y = \log_3 x$. Then $3^y = x$. Based on the second and fourth logarithm in the equation, $x \neq 1$ so $y \neq 0$. Therefore, we have $x^{1/y} = 3$ so that $1/y = \log_x 3$. We obtain that

$$\begin{aligned}
&\log_x 27 = \log_x(3^3) = 3/y,\\
&\log_3(9x^3) = 2\log_3 3 + 3\log_3 x = 2 + 3y,\\
&\log_x(9/x) = \log_x(3^2) + \log_x(x^{-1}) = 2/y - 1.
\end{aligned}$$

Now the equation can be rewritten as $y + 3/y = 3y + 2/y + 1$. After simplification, $2y^2 + y - 1 = 0$. This quadratic equation has two solutions, $y = -1$ and $y = 1/2$. Since both solutions are different from 0, it follows that $x = 3^{-1}$ and $x = 3^{1/2}$ are solutions of the original equation.

13. Find the exact value of

$$E = \sum_{k=1}^{8} \sin^6 \frac{(2k-1)\pi}{32}.$$

Answer: $\frac{5}{2}$

Solution: Using the formula $\sin^2\alpha = \dfrac{1 - \cos 2\alpha}{2}$ we have

$$E = \frac{1}{8}\left[8 - 3\sum_{k=1}^{8}\cos\frac{(2k-1)\pi}{16} + 3\sum_{k=1}^{8}\cos^2\frac{(2k-1)\pi}{16} - \sum_{k=1}^{8}\cos^3\frac{(2k-1)\pi}{16}\right].$$

Next we see that

$$\begin{aligned}\sum_{k=1}^{8}\cos\frac{(2k-1)\pi}{16} &= \cos\frac{\pi}{16}+\cos\frac{3\pi}{16}+\cos\frac{5\pi}{16}+\cos\frac{7\pi}{16}\\ &\quad+\cos\left(\pi-\frac{7\pi}{16}\right)+\cos\left(\pi-\frac{5\pi}{16}\right)+\cos\left(\pi-\frac{3\pi}{16}\right)+\cos\left(\pi-\frac{\pi}{16}\right)\\ &= \cos\frac{\pi}{16}+\cos\frac{3\pi}{16}+\cos\frac{5\pi}{16}+\cos\frac{7\pi}{16}-\cos\frac{7\pi}{16}-\cos\frac{5\pi}{16}-\cos\frac{3\pi}{16}-\cos\frac{\pi}{16}=0,\end{aligned}$$

and similarly, $\displaystyle\sum_{k=1}^{8}\cos^3\frac{(2k-1)\pi}{16}=0$. Therefore,

$$E=\frac{1}{8}\left[8+3\sum_{k=1}^{8}\cos^2\frac{(2k-1)\pi}{16}\right]=\frac{1}{8}\left[8+\frac{3}{2}\sum_{k=1}^{8}\left(1+\cos\frac{(2k-1)\pi}{8}\right)\right]=\frac{1}{8}(8+\frac{3}{2}\cdot 8)=\frac{5}{2},$$

since $\displaystyle\sum_{k=1}^{8}\cos\frac{(2k-1)\pi}{8}=0$.

14. Find the largest integer N such that

$$\sum_{n=1}^{2025}\frac{1}{n^3+3n^2+2n}<\frac{1}{N}.$$

Answer: 4

Solution: We can write

$$\frac{1}{n(n+1)(n+2)}=\frac{A}{n}+\frac{B}{n+1}+\frac{C}{n+2}$$

(this is a partial fraction decomposition). Getting a common denominator on the right gives us

$$\frac{1}{n(n+1)(n+2)}=\frac{A(n^2+3n+2)+B(n^2+2n)+C(n^2+n)}{n(n+1)(n+2)}$$

$$\frac{1}{n(n+1)(n+2)}=\frac{(A+B+C)n^2+(3A+2B+C)n+2A}{n(n+1)(n+2)}$$

Since there are no n terms in the numerator on the left, we need to solve the following system of equations:

$$\begin{gathered}A+B+C=0\\ 3A+2B+C=0\\ 1=2A\end{gathered}$$

meaning $A=\frac{1}{2}$, $B+C=-\frac{1}{2}$, and $2B+C=-\frac{3}{2}\Rightarrow A=\frac{1}{2}$, $B=-1$, and $C=\frac{1}{2}$. Therefore,

$$\sum_{n=1}^{2025}\frac{1}{n^3+3n^2+2n}=\sum_{n=1}^{2025}\frac{1}{2}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\left(\frac{1}{1} - \frac{2}{2} + \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} \cancel{- \frac{2}{3}} + \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{2}{4} + \frac{1}{5}\right) + \cdots + \left(\frac{1}{2025} - \frac{2}{2026} + \frac{1}{2027}\right)\right).$$

The cancellation shown continues throughout, leaving us

$$\frac{1}{2}\left(1 - \frac{2}{2} + \frac{1}{2} + \frac{1}{2026} - \frac{2}{2026} + \frac{1}{2027}\right) = \frac{1}{4} + \frac{1}{2}\cdot\left(\frac{1}{2027} - \frac{1}{2026}\right).$$

$$= \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{(2026)(2027)}$$

Since $\frac{1}{5} = \frac{1}{4} - \frac{1}{20} < \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{(2026)(2027)}$, we see that

$$\frac{1}{5} < \sum_{n=1}^{2025} \frac{1}{n^3 + 3n^2 + 2n} < \frac{1}{4}$$

So $N = 4$.

15. A function $f : \mathbb{N} \to \mathbb{N}$ on the set of positive integers is defined recursively as follows: $f(1) = 1$, $f(n) = f(n-1) + 1$ for all odd numbers $n \geq 3$, and $f(n) = f(n/2)$ for all even n. Find $f(2025)$.

Answer: 8.

The function $f(n)$ counts the number of 1s in the binary representation of the integer n. Recall that the binary representation of n is a string of symbols $d_s d_{s-1} \ldots d_3 d_2 d_1$, where each d_i is 0 or 1, $d_s \neq 0$, and $n = d_1 + 2d_2 + 2^2 d_3 + \cdots + 2^{s-2} d_{s-1} + 2^{s-1} d_s$. Let $g(n)$ denote the number of 1s in the binary representation of n. For any integer $k \geq 1$, the binary representation of the number $2k$ is obtained from the binary representation of k by appending 0 while the binary representation of $2k+1$ is obtained by appending 1. It follows that $g(2k) = g(k)$ and $g(2k+1) = g(k)+1$. Note that we also have $f(2k) = f(k)$ and $f(2k+1) = f(2k) + 1 = f(k) + 1$. Besides, $f(1) = g(1) = 1$. Since any integer $n \geq 2$ can be written as $2k$ or $2k+1$ for some $k \in \mathbb{N}$, it follows by strong induction on n that $f(n) = g(n)$ for all $n \in \mathbb{N}$.

To obtain the binary representation of the number 2025, observe that $2^{11} = 2048 > 2025$ and $2^{11} - 2^5 = 2016 < 2025$ so that $2025 = 2^{11} - 2^5 + 9$. The binary representation of $2^{11} - 2^5$ is 11111100000. The binary representation of 9 is 1001. It follows that the binary representation of 2025 is 11111101001. It contains eight 1s.

16. Evaluate the product $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}$.

Answer: $-\frac{1}{8}$

Solution: Using the trigonometric identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$, we obtain that

$$\cos\alpha = \frac{\sin 2\alpha}{2\sin\alpha}$$

whenever $\sin\alpha \neq 0$. It follows that

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = \frac{\sin(2\pi/7)}{2\sin(\pi/7)}\cdot\frac{\sin(6\pi/7)}{2\sin(3\pi/7)}\cdot\frac{\sin(10\pi/7)}{2\sin(5\pi/7)}.$$

Further, using the trigonometric identities $\sin(\pi-\alpha) = \sin\alpha$ and $\sin(\pi+\alpha) = -\sin\alpha$, we obtain that $\sin(2\pi/7) = \sin(5\pi/7)$, $\sin(6\pi/7) = \sin(\pi/7)$, and $\sin(10\pi/7) = -\sin(3\pi/7)$. Hence

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = \frac{\sin(5\pi/7)}{2\sin(\pi/7)}\cdot\frac{\sin(\pi/7)}{2\sin(3\pi/7)}\cdot\frac{-\sin(3\pi/7)}{2\sin(5\pi/7)} = -\frac{1}{8}.$$

17. Find all solutions of the equation

$$\sqrt{x+\sqrt{4x-3}+\frac{1}{4}}+\sqrt{x-\sqrt{4x-3}+\frac{1}{4}} = x.$$

Answer: 3

Solution: The equation is equivalent to $\sqrt{\left(\frac{\sqrt{4x-3}}{2}+1\right)^2}+\sqrt{\left(\frac{\sqrt{4x-3}}{2}-1\right)^2} = x$. So we get

$$\left|\frac{\sqrt{4x-3}}{2}+1\right|+\left|\frac{\sqrt{4x-3}}{2}-1\right| = x.$$

Note that the first absolute value term is positive whenever defined ($x \geq \frac{3}{4}$). For the second we have

$$\left|\frac{\sqrt{4x-3}}{2}-1\right| = \begin{cases} \frac{\sqrt{4x-3}}{2}-1, & \text{if } \frac{3}{4} \leq x < \frac{7}{4} \\ -\left(\frac{\sqrt{4x-3}}{2}-1\right), & \text{if } x \geq \frac{7}{4} \end{cases}$$

Thus, if $\frac{3}{4} \leq x < \frac{7}{4}$ the equation becomes $\frac{\sqrt{4x-3}}{2}+1-\frac{\sqrt{4x-3}}{2}+1 = x$, so $x = 2$, which is impossible. If $\frac{7}{4} \leq x$, then the equation becomes $\frac{\sqrt{4x-3}}{2}+1+\frac{\sqrt{4x-3}}{2}-1 = x$, so we need to solve $\sqrt{4x-3} = x$. Squaring both sides we get $4x-3 = x^2$, with solutions 3 and 1. Since $x \geq \frac{7}{4}$, only $x = 3$ is a solution.

18. Consider the trigonometric equation

$$(\sin 2x+\sqrt{3}\cos 2x)^2-5 = \cos\left(\frac{\pi}{6}-2x\right).$$

Find the sum of all solutions of this equation which lie in the interval $[0, 4\pi]$.

Answer: $\dfrac{100\pi}{12} = \dfrac{25\pi}{3}$

Solution: The above equation is equivalent to

$$\left(\frac{1}{2}\sin 2x+\frac{\sqrt{3}}{2}\cos 2x\right)^2-\frac{5}{4}=\frac{1}{4}\cos\left(\frac{\pi}{6}-2x\right)\Leftrightarrow$$
$$\cos^2\left(\frac{\pi}{6}-2x\right)-\frac{1}{4}\cos\left(\frac{\pi}{6}-2x\right)-\frac{5}{4}=0.$$

Let $y=\cos\left(\frac{\pi}{6}-2x\right)$. Then $y\in[-1,1]$ and $4y^2-y-5=0\Leftrightarrow(4y-5)(y+1)=0$ which implies that $y=-1$. So

$$\cos\left(\frac{\pi}{6}-2x\right)=-1\Leftrightarrow 2x-\frac{\pi}{6}=\pi+2n\pi, n\in\mathbb{Z}\Leftrightarrow x=\frac{7\pi}{12}+n\pi, n\in\mathbb{Z}.$$

The solutions in $[0,4\pi]$ are $\frac{7\pi}{12},\frac{19\pi}{12},\frac{31\pi}{12}$ and $\frac{43\pi}{12}$ and their sum is $\frac{100\pi}{12}=\frac{25\pi}{3}$.

19. Find $\left\lfloor\sqrt{2}^{\sqrt{3}}-\sqrt{3}^{\sqrt{2}}\right\rfloor$, that is, the largest integer not exceeding $\sqrt{2}^{\sqrt{3}}-\sqrt{3}^{\sqrt{2}}$.

Answer: -1

Solution: We are going to show that $1.5<\sqrt{2}^{\sqrt{3}}<2$ and $2<\sqrt{3}^{\sqrt{2}}<2.5$. This will imply that $-1<\sqrt{2}^{\sqrt{3}}-\sqrt{3}^{\sqrt{2}}<0$ so that $\left\lfloor\sqrt{2}^{\sqrt{3}}-\sqrt{3}^{\sqrt{2}}\right\rfloor=-1$.
All our estimates are based on the fact that inequalities $0<x<y$ imply that $a^x<a^y$ for any $a>1$ and $x^b<y^b$ for any $b>0$. Let us begin with the upper estimates. We obtain that $\sqrt{2}^{\sqrt{3}}<\sqrt{2}^{\sqrt{4}}=\sqrt{2}^2=2$ and

$$\sqrt{3}^{\sqrt{2}}<\sqrt{3}^{\sqrt{9/4}}=\left(3^{1/2}\right)^{3/2}=3^{3/4}=\sqrt[4]{3^3}=\sqrt[4]{27}<\sqrt[4]{36}=\sqrt[4]{6^2}=\sqrt{6}<\sqrt{\frac{25}{4}}=\frac{5}{2}=2.5.$$

Further,

$$\sqrt{2}^{\sqrt{3}}>\sqrt{2}^{\sqrt{25/9}}=\left(2^{1/2}\right)^{5/3}=2^{5/6}=\sqrt[6]{2^5}=\sqrt[6]{32}>\sqrt[6]{27}=\sqrt[6]{3^3}=\sqrt{3}>\sqrt{\frac{9}{4}}=\frac{3}{2}=1.5.$$

To obtain the last estimate, we first observe that $3^7=2187>1024=2^{10}$. It follows that $3^{7/10}>2$. Then

$$\sqrt{3}^{\sqrt{2}}=\left(3^{1/2}\right)^{\sqrt{2}}=3^{\sqrt{2}/2}=3^{\sqrt{1/2}}>3^{\sqrt{49/100}}=3^{7/10}>2.$$

20. Find all the triples of positive real numbers (x,y,z) that satisfy the equations

$$\frac{4\sqrt{x^2+1}}{x}=\frac{5\sqrt{y^2+1}}{y}=\frac{6\sqrt{z^2+1}}{z},$$
$$x+y+z=xyz.$$

Answer: $\left(\frac{\sqrt{7}}{3},\frac{5\sqrt{7}}{9},3\sqrt{7}\right)$ or $x=\frac{\sqrt{7}}{3}, y=\frac{5\sqrt{7}}{9}, z=3\sqrt{7}$

Solution: Since x, y and z are positive real numbers, there exist A, B, and C in $\left(0, \dfrac{\pi}{2}\right)$ such that $x = \tan A$, $y = \tan B$ and $z = \tan C$. The last equation becomes

$$\begin{aligned}
&\tan A + \tan B + \tan C = \tan A \tan B \tan C \Leftrightarrow \\
&\sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C = 0 \Leftrightarrow \\
&\sin(A+B)\cos C + \cos(A+B)\sin C = 0 \Leftrightarrow \sin(A+B+C) = 0.
\end{aligned}$$

Since $A, B, C \in \left(0, \dfrac{\pi}{2}\right)$, we obtain that $A+B+C = \pi$, which says that A, B, and C are the angles of a triangle. Using the identity $\sin\alpha = \dfrac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$, for all $\alpha \in \left(0, \dfrac{\pi}{2}\right)$, the first equations of our system become

$$\frac{4}{\sin A} = \frac{5}{\sin B} = \frac{6}{\sin C}.$$

From the Law of Sines, we can assume that $a = 4$, $b = 5$, and $c = 6$. We apply the Law of Cosines and we get

$$\begin{aligned}
&\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{3}{4} \Rightarrow \sin A = \frac{\sqrt{7}}{4} \Rightarrow x = \tan A = \frac{\sqrt{7}}{3}, \\
&\cos B = \frac{9}{16} \Rightarrow \sin B = \frac{5\sqrt{7}}{16} \Rightarrow y = \tan B = \frac{5\sqrt{7}}{9}, \\
&\cos C = \frac{1}{8} \Rightarrow \sin C = \frac{3\sqrt{7}}{8} \Rightarrow z = \tan C = 3\sqrt{7}.
\end{aligned}$$