

Solutions to EF Exam
Texas A&M High School Math Contest
25 October, 2025

1. A rectangle has perimeter 17 and area 15. Find the length of the diagonal of this rectangle.

Answer: $\frac{13}{2} = 6.5$.

Let a and b be lengths of two adjacent sides of the rectangle. Then the rectangle has perimeter $2(a + b)$ and area ab , and its diagonals have length $\sqrt{a^2 + b^2}$. We are given that $2(a + b) = 17$ and $ab = 15$. It follows that $b = 15/a$ and $2(a + 15/a) = 17$. Likewise, $2(b + 15/b) = 17$. Hence a and b are solutions of the equation $2(x + 15/x) = 17$. This equation is equivalent to a quadratic equation $2x^2 - 17x + 30 = 0$, which has solutions $5/2$ and 6 . Note that $(5/2)^2 \neq 15$ and $6^2 \neq 15$. Hence one of the lengths a and b is $5/2$ and the other is 6 . Then the diagonals of the rectangle have length $\sqrt{(5/2)^2 + 6^2} = \sqrt{169/4} = 13/2$.

2. Let f be a one-to-one differentiable function and let g denote its inverse. Find $h'(3)$ if $h(x) = x^2g(x)$, $f(2) = 3$, and $f'(2) = 7$.

Answer: $\frac{93}{7}$

Solution: By the Product Rule we get

$$h'(x) = 2xg(x) + x^2g'(x) \Rightarrow h'(3) = 6g(3) + 9g'(3).$$

Since $f(2) = 3$ and $f'(2) = 7$, we have that $g(3) = 2$ and

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(2)} = \frac{1}{7}.$$

Therefore, $h'(3) = 12 + \frac{9}{7} = \frac{93}{7}$.

3. The expression $\left(x + 1 + \frac{1}{x}\right)^4$ can be expanded as a sum $\sum_{k=-4}^4 c_k x^k$, where each c_k is a real number.

Find the constant term c_0 .

Answer: 19.

The expanded expression is the sum of $3^4 = 81$ products of the form $y_1 y_2 y_3 y_4$, where each of the factors y_i can be chosen to be either x or 1 or $\frac{1}{x}$. Every product is equal to x^k for some integer k . Hence c_0 counts the number of choices when the product $y_1 y_2 y_3 y_4$ equals 1. We have three different ways to arrange this. First we can choose all four factors to be 1. Secondly, we can choose one of the four factors to be x , one of the remaining three factors to be $\frac{1}{x}$, and the other two factors to

be 1. There are $4 \cdot 3 = 12$ different choices here. Finally, we can choose two of the four factors to be x and the other two factors to be $\frac{1}{x}$. There are $\binom{4}{2} = \frac{4 \cdot 3}{2!} = 6$ different choices here. Thus $c_0 = 1 + 12 + 6 = 19$.

4. Find a real solution x of the equation $\sqrt{x\sqrt{x\sqrt{x}}} = 8\sqrt{2}$.

Answer: 16.

Clearly, any real solution should be nonnegative. For any $x \geq 0$ we obtain

$$\begin{aligned}\sqrt{x} &= x^{1/2}, \\ x\sqrt{x} &= x \cdot x^{1/2} = x^{3/2}, \\ \sqrt{x\sqrt{x}} &= (x^{3/2})^{1/2} = x^{3/4}, \\ x\sqrt{x\sqrt{x}} &= x \cdot x^{3/4} = x^{7/4}, \\ \sqrt{x\sqrt{x\sqrt{x}}} &= (x^{7/4})^{1/2} = x^{7/8}.\end{aligned}$$

Hence the given equation is equivalent to $x^{7/8} = 8\sqrt{2}$. We observe that

$$8\sqrt{2} = 2^3 \cdot 2^{1/2} = 2^{7/2} = (2^4)^{7/8} = 16^{7/8}.$$

Since the function $f(x) = x^{7/8}$ is strictly increasing on the interval $[0, \infty)$, it follows that $x = 16$ is the only real solution of the equation.

5. Find the limit of the sequence $a_n = \sin^2(\pi\sqrt{n^2 + n + 1})$.

Answer: 1

Solution: We have

$$\begin{aligned}\lim_{n \rightarrow \infty} \sin^2(\pi\sqrt{n^2 + n + 1}) &= \lim_{n \rightarrow \infty} [\sin(\pi\sqrt{n^2 + n + 1} - n\pi + n\pi)]^2 \\ &= \lim_{n \rightarrow \infty} [(-1)^n \sin(\pi\sqrt{n^2 + n + 1} - n\pi)]^2 = \lim_{n \rightarrow \infty} \left[\sin \left(\pi \frac{n^2 + n + 1 - n^2}{\sqrt{n^2 + n + 1} + n} \right) \right]^2 = \sin^2 \frac{\pi}{2} = 1.\end{aligned}$$

6. A polynomial $f(x) = x^5 - 2x^4 + ax^2 + bx$, where a and b are unknown coefficients, is divisible by $x^2 - 3x + 2$. Find ab .

Answer: -2.

The polynomial $x^2 - 3x + 2$ has roots 1 and 2. Since the polynomial $f(x)$ is divisible by it, we have $f(x) = (x^2 - 3x + 2)q(x)$ for some polynomial $q(x)$. It follows that 1 and 2 are also roots of $f(x)$. The equalities $f(1) = f(2) = 0$ give rise to a system of linear equations in variables a and b :

$$\begin{cases} -1 + a + b = 0 \\ 4a + 2b = 0 \end{cases} \iff \begin{cases} a + b = 1 \\ 2a + b = 0 \end{cases} \iff \begin{cases} a = -1 \\ b = 2 \end{cases} \iff ab = -2.$$

Thus $ab = -1 \cdot 2 = -2$.

7. Evaluate $\lim_{n \rightarrow \infty} \left(\frac{4}{3} \sum_{k=1}^n \frac{1}{k(k+2)} \right)^n$.

Answer: $e^{-\frac{4}{3}}$

Solution: We first see that

$$\frac{1}{k(k+2)} = \frac{1}{2k} - \frac{1}{2(k+2)}$$

for every positive integer k , which implies that the sequence $\{x_n\}$, where

$$x_n = \sum_{k=1}^n \frac{1}{k(k+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

has limit $\frac{3}{4}$. Our limit then becomes

$$\lim_{n \rightarrow \infty} \left(\frac{4}{3} x_n \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{4}{3} x_n - 1 \right)^n = e^{-\frac{4}{3}},$$

since $\lim_{x \rightarrow 0} (1+x)^{\frac{a}{x}} = e^a$, $\lim_{n \rightarrow \infty} \frac{4}{3} x_n - 1 = 0$, and

$$\lim_{n \rightarrow \infty} n \left(\frac{4}{3} x_n - 1 \right) = \lim_{n \rightarrow \infty} n \left(-\frac{2}{3(n+1)} - \frac{2}{3(n+2)} \right) = -\frac{4}{3}.$$

8. Let f be a continuous function such that $\int_{-1}^1 f(x^2) dx = 2025$. Find $I = \int_{-1}^1 \frac{f(x^2)}{5^x + 1} dx$.

Answer: $\frac{2025}{2}$

Solution: We make the substitution $u = -x$ and we get

$$\begin{aligned} I &= - \int_1^{-1} \frac{f(u^2)}{5^{-u} + 1} du = \int_{-1}^1 \frac{5^u f(u^2)}{5^u + 1} du \\ &= \int_{-1}^1 \frac{(5^u + 1) f(u^2)}{5^u + 1} du - \int_{-1}^1 \frac{f(u^2)}{5^u + 1} du = 2025 - I. \end{aligned}$$

Therefore, $I = \frac{2025}{2}$.

9. Find the x -coordinate of all the points on the graph of $f(x) = \frac{x}{x+1}$ where the tangent line also passes through the point $(1, 2)$.

Answer: $-2 \pm \sqrt{3}$

Solution: On one hand, the slope of the tangent line to the graph of f at the point $(a, \frac{a}{a+1})$ that passes through $(1, 2)$ is equal to

$$\frac{\frac{a}{a+1} - 2}{a - 1} = \frac{a - 2(a+1)}{(a+1)(a-1)} = -\frac{a+2}{(a+1)(a-1)}.$$

On the other hand, the same slope is equal to $f'(a) = \frac{1}{(a+1)^2}$. Therefore,

$$-\frac{a+2}{(a+1)(a-1)} = \frac{1}{(a+1)^2} \Leftrightarrow (a+2)(a+1) = 1-a \Leftrightarrow a^2 + 4a + 1 = 0.$$

Solving the quadratic equation we get $a = -2 - \sqrt{3}$ or $a = -2 + \sqrt{3}$.

10. Suppose T is a triangle such that the center of its circumscribed circle lies within the triangle. Let α , β and γ be the angles of T . Find the largest real number c such that the inequality

$$\sin \alpha + \sin \beta + \sin \gamma \geq c$$

is guaranteed for any choice of the triangle T .

Answer: 2.

Without loss of generality, we may assume that $\alpha \geq \beta \geq \gamma$. Using the formula for the sum of sines and the identity $\alpha + \beta + \gamma = \pi$, we obtain

$$\sin \alpha + \sin \gamma = 2 \sin \frac{\alpha + \gamma}{2} \cos \frac{\alpha - \gamma}{2} = 2 \sin \frac{\pi - \beta}{2} \cos \frac{\alpha - \gamma}{2}.$$

Let $\alpha_0 = \pi/2$ and $\gamma_0 = \pi/2 - \beta$. Then

$$\sin \alpha_0 + \sin \gamma_0 = 2 \sin \frac{\alpha_0 + \gamma_0}{2} \cos \frac{\alpha_0 - \gamma_0}{2} = 2 \sin \frac{\pi - \beta}{2} \cos \frac{\beta}{2}.$$

The center of the circumscribed circle lies outside a triangle if and only if the triangle is obtuse. Hence $\alpha \leq \pi/2$. It follows that $\beta + \gamma \geq \pi/2 \geq \alpha$. Then $\alpha - \gamma \leq \beta$. Consequently, $\cos \frac{\alpha - \gamma}{2} \geq \cos \frac{\beta}{2}$, which implies that $\sin \alpha + \sin \gamma \geq \sin \alpha_0 + \sin \gamma_0 = 1 + \sin \gamma_0$. The angles β and γ_0 are both acute. Therefore their sines and cosines belong to the interval $(0, 1)$. It follows that

$$\sin \gamma_0 + \sin \beta > \sin \gamma_0 \cos \beta + \cos \gamma_0 \sin \beta = \sin(\gamma_0 + \beta) = \sin(\pi/2) = 1.$$

Finally, $\sin \alpha + \sin \beta + \sin \gamma \geq 1 + \sin \gamma_0 + \sin \beta > 2$.

To prove that $c = 2$ is the largest number with the required property, it is enough to show that the sum $\sin \alpha + \sin \beta + \sin \gamma$ can be arbitrarily close to 2. Indeed, let us choose T to be an acute triangle with two larger angles α and β being close to $\pi/2$. Then the third angle γ is close to 0. α and β can be chosen arbitrarily close to $\pi/2$ while γ is arbitrarily close to 0. That way $\sin \alpha + \sin \beta + \sin \gamma$ can be made arbitrarily close to 2.

11. Determine how many real solutions (x, y) the following system of equations has:

$$\begin{cases} x^2 + y^2 = 18, \\ \sin(x - y) = 0. \end{cases}$$

Answer: 6.

The equation $\sin z = 0$ has solutions $z = \pi n$, where n is an arbitrary integer. Therefore $x - y = \pi n$, $n \in \mathbb{Z}$. Note that

$$(x + y)^2 + (x - y)^2 = (x^2 + 2xy + y^2) + (x^2 - 2xy + y^2) = 2(x^2 + y^2).$$

Hence the condition $x^2 + y^2 = 18$ implies that $(x - y)^2 \leq 36$. Since $(2\pi)^2 > 6^2 = 36$, it follows that $x - y = 0$ or π or $-\pi$. In the case $x - y = 0$, we obtain two solutions of the system: $x = y = 3$ and $x = y = -3$.

In the case $x - y = \pi$, let us substitute $y + \pi$ for x into the first equation of the system: $(y + \pi)^2 + y^2 = 18$. After simplification, $2y^2 + 2\pi y + (\pi^2 - 18) = 0$. Since $\pi^2 - 18 < 0$, it is clear that the quadratic equation has two real solutions y_1 and y_2 . This yields two solutions of the system: $(y_1 + \pi, y_1)$ and $(y_2 + \pi, y_2)$. As for the case $x - y = -\pi$, we notice that whenever (x_0, y_0) is a solution of the system, so is (y_0, x_0) . It follows that the only solutions of the system in this case are $(y_1, \pi + y_1)$ and $(y_2, \pi + y_2)$. Thus the system has a total of six real solutions.

12. Evaluate $\lim_{n \rightarrow \infty} a_n$, where

$$a_n = \frac{\sin\left(1 + \frac{1}{n}\right)}{\sqrt{n^2 + 1}} + \frac{\sin\left(1 + \frac{2}{n}\right)}{\sqrt{n^2 + 2}} + \cdots + \frac{\sin\left(1 + \frac{n}{n}\right)}{\sqrt{n^2 + n}}.$$

Answer: $\cos 1 - \cos 2$

Solution: Let $\{b_n\}$ and $\{c_n\}$ be sequences defined by

$$b_n = \frac{1}{\sqrt{n^2 + n}} \left[\sin\left(1 + \frac{1}{n}\right) + \sin\left(1 + \frac{2}{n}\right) + \cdots + \sin\left(1 + \frac{n}{n}\right) \right]$$

and

$$c_n = \frac{1}{\sqrt{n^2 + 1}} \left[\sin\left(1 + \frac{1}{n}\right) + \sin\left(1 + \frac{2}{n}\right) + \cdots + \sin\left(1 + \frac{n}{n}\right) \right].$$

We have that $b_n \leq a_n \leq c_n$, for all positive integers n . If we denote

$$d_n = \frac{1}{n} \left[\sin\left(1 + \frac{1}{n}\right) + \sin\left(1 + \frac{2}{n}\right) + \cdots + \sin\left(1 + \frac{n}{n}\right) \right]$$

then we see that

$$\lim_{n \rightarrow \infty} d_n = \int_0^1 \sin(1 + x) dx = -\cos(1 + x) \Big|_0^1 = \cos 1 - \cos 2.$$

Since $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} d_n$, by the Squeeze Theorem we get that $\lim_{n \rightarrow \infty} a_n = \cos 1 - \cos 2$.

13. Find all solutions x of the equation $\log_3 x + \log_x 27 = \log_3(9x^3) + \log_x(9/x)$.

Answer: $\frac{1}{3}; \sqrt{3}$.

Let us introduce a new variable $y = \log_3 x$. Then $3^y = x$. Based on the second and fourth logarithm in the equation, $x \neq 1$ so $y \neq 0$. Therefore, we have $x^{1/y} = 3$ so that $1/y = \log_x 3$. We obtain that

$$\begin{aligned}\log_x 27 &= \log_x(3^3) = 3/y, \\ \log_3(9x^3) &= 2\log_3 3 + 3\log_3 x = 2 + 3y, \\ \log_x(9/x) &= \log_x(3^2) + \log_x(x^{-1}) = 2/y - 1.\end{aligned}$$

Now the equation can be rewritten as $y + 3/y = 3y + 2/y + 1$. After simplification, $2y^2 + y - 1 = 0$. This quadratic equation has two solutions, $y = -1$ and $y = 1/2$. Since both solutions are different from 0, it follows that $x = 3^{-1}$ and $x = 3^{1/2}$ are solutions of the original equation.

14. Find the exact value of

$$E = \sum_{k=1}^8 \sin^6 \frac{(2k-1)\pi}{32}.$$

Answer: $\frac{5}{2}$

Solution: Using the formula $\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$ we have

$$E = \frac{1}{8} \left[8 - 3 \sum_{k=1}^8 \cos \frac{(2k-1)\pi}{16} + 3 \sum_{k=1}^8 \cos^2 \frac{(2k-1)\pi}{16} - \sum_{k=1}^8 \cos^3 \frac{(2k-1)\pi}{16} \right].$$

Next we see that

$$\begin{aligned}\sum_{k=1}^8 \cos \frac{(2k-1)\pi}{16} &= \cos \frac{\pi}{16} + \cos \frac{3\pi}{16} + \cos \frac{5\pi}{16} + \cos \frac{7\pi}{16} \\ &\quad + \cos \left(\pi - \frac{7\pi}{16} \right) + \cos \left(\pi - \frac{5\pi}{16} \right) + \cos \left(\pi - \frac{3\pi}{16} \right) + \cos \left(\pi - \frac{\pi}{16} \right) \\ &= \cos \frac{\pi}{16} + \cos \frac{3\pi}{16} + \cos \frac{5\pi}{16} + \cos \frac{7\pi}{16} - \cos \frac{7\pi}{16} - \cos \frac{5\pi}{16} - \cos \frac{3\pi}{16} - \cos \frac{\pi}{16} = 0,\end{aligned}$$

and similarly, $\sum_{k=1}^8 \cos^3 \frac{(2k-1)\pi}{16} = 0$. Therefore,

$$E = \frac{1}{8} \left[8 + 3 \sum_{k=1}^8 \cos^2 \frac{(2k-1)\pi}{16} \right] = \frac{1}{8} \left[8 + \frac{3}{2} \sum_{k=1}^8 \left(1 + \cos \frac{(2k-1)\pi}{8} \right) \right] = \frac{1}{8} (8 + \frac{3}{2} \cdot 8) = \frac{5}{2},$$

since $\sum_{k=1}^8 \cos \frac{(2k-1)\pi}{8} = 0$.

15. Evaluate the integral $\int_0^{\sqrt{3}} \arcsin\left(\frac{2x}{x^2+1}\right) dx$.

Answer: $\frac{\pi\sqrt{3}}{3}$

Solution: If we integrate by parts we get

$$\int_0^1 \arcsin\left(\frac{2x}{x^2+1}\right) dx = x \arcsin\left(\frac{2x}{x^2+1}\right) \Big|_0^1 - \int_0^1 \frac{2x}{x^2+1} dx,$$

since $\frac{d}{dx} \left(\arcsin \frac{2x}{x^2+1} \right) = \frac{2}{x^2+1}$, for all $x \in (-1, 1)$. This implies that

$$\int_0^1 \arcsin\left(\frac{2x}{x^2+1}\right) dx = \frac{\pi}{2} - \ln(x^2+1) \Big|_0^1 = \frac{\pi}{2} - \ln 2.$$

Similarly,

$$\int_1^{\sqrt{3}} \arcsin\left(\frac{2x}{x^2+1}\right) dx = x \arcsin\left(\frac{2x}{x^2+1}\right) \Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{2x}{x^2+1} dx,$$

since $\frac{d}{dx} \left(\arcsin \frac{2x}{x^2+1} \right) = -\frac{2}{x^2+1}$, for all $x \in (1, \infty)$, which implies that

$$\int_1^{\sqrt{3}} \arcsin\left(\frac{2x}{x^2+1}\right) dx = \frac{\pi\sqrt{3}}{3} - \frac{\pi}{2} + \ln(x^2+1) \Big|_1^{\sqrt{3}} = \frac{\pi\sqrt{3}}{3} - \frac{\pi}{2} + 2\ln 2 - \ln 2 = \frac{\pi\sqrt{3}}{3} - \frac{\pi}{2} + \ln 2.$$

Therefore, $\int_0^{\sqrt{3}} \arcsin \frac{2x}{x^2+1} dx = \frac{\pi\sqrt{3}}{3}$.

16. A function $f : \mathbb{N} \rightarrow \mathbb{N}$ on the set of positive integers is defined recursively as follows: $f(1) = 1$, $f(n) = f(n-1) + 1$ for all odd numbers $n \geq 3$, and $f(n) = f(n/2)$ for all even n . Find $f(2025)$.

Answer: 8.

The function $f(n)$ counts the number of 1s in the binary representation of the integer n . Recall that the binary representation of n is a string of symbols $d_s d_{s-1} \dots d_3 d_2 d_1$, where each d_i is 0 or 1, $d_s \neq 0$, and $n = d_1 + 2d_2 + 2^2 d_3 + \dots + 2^{s-2} d_{s-1} + 2^{s-1} d_s$. Let $g(n)$ denote the number of 1s in the binary representation of n . For any integer $k \geq 1$, the binary representation of the number $2k$ is obtained from the binary representation of k by appending 0 while the binary representation of $2k+1$ is obtained by appending 1. It follows that $g(2k) = g(k)$ and $g(2k+1) = g(k) + 1$. Note that we also have $f(2k) = f(k)$ and $f(2k+1) = f(2k) + 1 = f(k) + 1$. Besides, $f(1) = g(1) = 1$. Since any integer $n \geq 2$ can be written as $2k$ or $2k+1$ for some $k \in \mathbb{N}$, it follows by strong induction on n that $f(n) = g(n)$ for all $n \in \mathbb{N}$.

To obtain the binary representation of the number 2025, observe that $2^{11} = 2048 > 2025$ and $2^{11} - 2^5 = 2016 < 2025$ so that $2025 = 2^{11} - 2^5 + 9$. The binary representation of $2^{11} - 2^5$ is 11111100000. The binary representation of 9 is 1001. It follows that the binary representation of 2025 is 11111101001. It contains eight 1s.

17. Consider the trigonometric equation

$$(\sin 2x + \sqrt{3} \cos 2x)^2 - 5 = \cos \left(\frac{\pi}{6} - 2x \right).$$

Find the sum of all solutions of this equation which lie in the interval $[0, 4\pi]$.

Answer: $\frac{100\pi}{12} = \frac{25\pi}{3}$

Solution: The above equation is equivalent to

$$\begin{aligned} \left(\frac{1}{2} \sin 2x + \frac{\sqrt{3}}{2} \cos 2x \right)^2 - \frac{5}{4} &= \frac{1}{4} \cos \left(\frac{\pi}{6} - 2x \right) \Leftrightarrow \\ \cos^2 \left(\frac{\pi}{6} - 2x \right) - \frac{1}{4} \cos \left(\frac{\pi}{6} - 2x \right) - \frac{5}{4} &= 0. \end{aligned}$$

Let $y = \cos \left(\frac{\pi}{6} - 2x \right)$. Then $y \in [-1, 1]$ and $4y^2 - y - 5 = 0 \Leftrightarrow (4y - 5)(y + 1) = 0$ which implies that $y = -1$. So

$$\cos \left(\frac{\pi}{6} - 2x \right) = -1 \Leftrightarrow 2x - \frac{\pi}{6} = \pi + 2n\pi, n \in \mathbb{Z} \Leftrightarrow x = \frac{7\pi}{12} + n\pi, n \in \mathbb{Z}.$$

The solutions in $[0, 4\pi]$ are $\frac{7\pi}{12}, \frac{19\pi}{12}, \frac{31\pi}{12}$ and $\frac{43\pi}{12}$ and their sum is $\frac{100\pi}{12} = \frac{25\pi}{3}$.

18. Evaluate the limit

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\binom{n}{2}} + \sqrt{\binom{n+1}{2}} + \cdots + \sqrt{\binom{2n}{2}}}{n^2}.$$

Answer: $\frac{3\sqrt{2}}{4}$ or $\frac{3}{2\sqrt{2}}$

Solution 1: Let

$$S_n = \sum_{k=0}^n \sqrt{\binom{n+k}{2}} = \sum_{k=0}^n \sqrt{\frac{(n+k)(n+k-1)}{2}} = \frac{1}{\sqrt{2}} \sum_{k=0}^n \sqrt{(n+k)(n+k-1)}.$$

We use the Squeeze theorem with the following bounds:

$$(n+k-1)^2 \leq (n+k)(n+k-1) \leq (n+k)^2.$$

Consider L_n and U_n defined by

$$L_n = \frac{1}{\sqrt{2}} \sum_{k=0}^n \sqrt{(n+k-1)^2} = \frac{1}{\sqrt{2}} \sum_{k=0}^n (n+k-1) \quad \text{and} \quad U_n = \frac{1}{\sqrt{2}} \sum_{k=0}^n (n+k).$$

We have

$$\sqrt{2}L_n = \sum_{k=0}^n (n+k-1) = n(n+1) + \frac{(n-2)(n+1)}{2} = \frac{(3n-2)(n+1)}{2},$$

which implies

$$\lim_{n \rightarrow \infty} \frac{L_n}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2}} \frac{(3n-2)(n+1)}{2n^2} = \frac{3}{2\sqrt{2}}.$$

Similarly, we have

$$\sqrt{2}U_n = \frac{1}{\sqrt{2}} \sum_{k=0}^n (n+k) = n(n+1) + \frac{n(n+1)}{2} = \frac{3n(n+1)}{2},$$

and

$$\lim_{n \rightarrow \infty} \frac{U_n}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2}} \frac{3n(n+1)}{2n^2} = \frac{3}{2\sqrt{2}}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{L_n}{n^2} \leq \lim_{n \rightarrow \infty} \frac{S_n}{n^2} \leq \lim_{n \rightarrow \infty} \frac{U_n}{n^2}.$$

The given limit becomes

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\binom{n}{2}} + \sqrt{\binom{n+1}{2}} + \cdots + \sqrt{\binom{2n}{2}}}{n^2} = \lim_{n \rightarrow \infty} \frac{S_n}{n^2} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}.$$

Solution 2: We apply Stolz-Cesaro Theorem (effectively, a discrete version of L'Hospital's Rule for sequences) with general terms $a_n = \sum_{k=0}^n \sqrt{\binom{n+k}{2}}$ and $b_n = n^2$, defined for all positive integers n . Since $\{b_n\}$ is strictly increasing and unbounded and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} &= \lim_{n \rightarrow \infty} \frac{\sqrt{\binom{2n+1}{2}} + \sqrt{\binom{2n+2}{2}} - \sqrt{\binom{n}{2}}}{(n+1)^2 - n^2} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{2n(2n+1)} + \sqrt{(2n+1)(2n+2)} - \sqrt{n(n-1)}}{\sqrt{2}(2n+1)} = \frac{3\sqrt{2}}{4}, \end{aligned}$$

it implies that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{3\sqrt{2}}{4}$.

19. Find the exact value of $E = \cot^6\left(\frac{\pi}{9}\right) + \cot^6\left(\frac{2\pi}{9}\right) + \cot^6\left(\frac{4\pi}{9}\right)$.

Answer: 433

Solution: Using the formula $\cot 3\alpha = \frac{\cot^3 \alpha - 3 \cot \alpha}{3 \cot^2 \alpha - 1}$ we get that

$$\cot^2 3\alpha = \frac{\cot^6 \alpha - 6 \cot^4 \alpha + 9 \cot^2 \alpha}{9 \cot^4 \alpha - 6 \cot^2 \alpha + 1}.$$

For $\alpha = \frac{\pi}{9}$, $\alpha = \frac{2\pi}{9}$ and $\alpha = \frac{4\pi}{9}$, we have $\cot^2 3\alpha = \frac{1}{3}$. We deduce that $x_1 = \cot^2 \frac{\pi}{9}$, $x_2 = \cot^2 \frac{2\pi}{9}$ and $x_3 = \cot^2 \frac{4\pi}{9}$ are the roots of the equation

$$\frac{x^3 - 6x^2 + 9x}{9x^2 - 6x + 1} = \frac{1}{3} \Leftrightarrow 3x^3 - 27x^2 + 33x - 1 = 0.$$

So $x_1 + x_2 + x_3 = 9$, $x_1x_2 + x_2x_3 + x_3x_1 = 11$, and $x_1x_2x_3 = \frac{1}{3}$. This implies that $x_1^2 + x_2^2 + x_3^2 = 59$ and from $3x_i^3 - 27x_i^2 + 33x_i - 1 = 0$, for $i = 1, 2, 3$ we obtain that

$$3(x_1^3 + x_2^3 + x_3^3) - 27(x_1^2 + x_2^2 + x_3^2) + 33(x_1 + x_2 + x_3) - 3 = 0 \Leftrightarrow x_1^3 + x_2^3 + x_3^3 = 9(x_1^2 + x_2^2 + x_3^2) - 11(x_1 + x_2 + x_3) + 1.$$

Therefore, $E = 9 \cdot 59 - 11 \cdot 9 + 1 = 433$.

20. Consider the sequences $\{a_n\}$ and $\{b_n\}$ given by

$$a_n = \sum_{k=1}^n \frac{3k+1}{k+1} \binom{2k}{k}$$

and $b_n = \sqrt[n]{a_n + 2}$. Find $\lim_{n \rightarrow \infty} b_n$.

Answer: 4

Solution: We have that

$$\begin{aligned} \binom{2k+2}{k+1} - \binom{2k}{k} &= \frac{(2k+2)!}{[(k+1)!]^2} - \frac{(2k)!}{(k!)^2} \\ &= \frac{(2k)!}{(k!)^2} \left[\frac{(2k+1)(2k+2)}{(k+1)^2} - 1 \right] = \frac{3k+1}{k+1} \binom{2k}{k}. \end{aligned}$$

Therefore,

$$\begin{aligned} a_n &= \sum_{k=1}^n \frac{3k+1}{k+1} \binom{2k}{k} = \sum_{k=1}^n \left(\binom{2k+2}{k+1} - \binom{2k}{k} \right) \\ &= \binom{2n+2}{n+1} - \binom{2}{1} = \binom{2n+2}{n+1} - 2, \end{aligned}$$

which implies that $b_n = \sqrt[n]{a_n + 2} = \sqrt[n]{\binom{2n+2}{n+1}}$.

Next we use the fact that, if a sequence $\{x_n\}$ satisfies $x_n > 0$, for all n and $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n}$ exists then

$\lim_{n \rightarrow \infty} \sqrt[n]{x_n} = \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n}$. Since

$$\lim_{n \rightarrow \infty} \frac{\binom{2(n+1)+2}{(n+1)+1}}{\binom{2n+2}{n+1}} = \lim_{n \rightarrow \infty} \frac{\binom{2n+4}{n+2}}{\binom{2n+2}{n+1}} = \lim_{n \rightarrow \infty} \frac{(2n+3)(2n+4)}{(n+2)^2} = 4,$$

we conclude that $\lim_{n \rightarrow \infty} b_n = 4$.

21. Find all the triples of positive real numbers (x, y, z) that satisfy the equations

$$\frac{4\sqrt{x^2+1}}{x} = \frac{5\sqrt{y^2+1}}{y} = \frac{6\sqrt{z^2+1}}{z},$$

$$x + y + z = xyz.$$

Answer: $\left(\frac{\sqrt{7}}{3}, \frac{5\sqrt{7}}{9}, 3\sqrt{7}\right)$ or $x = \frac{\sqrt{7}}{3}, y = \frac{5\sqrt{7}}{9}, z = 3\sqrt{7}$

Solution: Since x, y and z are positive real numbers, there exist A, B , and C in $\left(0, \frac{\pi}{2}\right)$ such that $x = \tan A$, $y = \tan B$ and $z = \tan C$. The last equation becomes

$$\begin{aligned}\tan A + \tan B + \tan C &= \tan A \tan B \tan C \Leftrightarrow \\ \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C &= 0 \Leftrightarrow \\ \sin(A+B) \cos C + \cos(A+B) \sin C &= 0 \Leftrightarrow \sin(A+B+C) = 0.\end{aligned}$$

Since $A, B, C \in \left(0, \frac{\pi}{2}\right)$, we obtain that $A+B+C = \pi$, which says that A, B , and C are the angles of a triangle. Using the identity $\sin \alpha = \frac{\tan \alpha}{\sqrt{1+\tan^2 \alpha}}$, for all $\alpha \in \left(0, \frac{\pi}{2}\right)$, the first equations of our system become

$$\frac{4}{\sin A} = \frac{5}{\sin B} = \frac{6}{\sin C}.$$

From the Law of Sines, we can assume that $a = 4$, $b = 5$, and $c = 6$. We apply the Law of Cosines and we get

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} = \frac{3}{4} \Rightarrow \sin A = \frac{\sqrt{7}}{4} \Rightarrow x = \tan A = \frac{\sqrt{7}}{3}, \\ \cos B &= \frac{9}{16} \Rightarrow \sin B = \frac{5\sqrt{7}}{16} \Rightarrow y = \tan B = \frac{5\sqrt{7}}{9}, \\ \cos C &= \frac{1}{8} \Rightarrow \sin C = \frac{3\sqrt{7}}{8} \Rightarrow z = \tan C = 3\sqrt{7}.\end{aligned}$$